# ELEMENTS
OF
# DESCRIPTIVE GEOMETRY

CHARLES E. FERRIS

come in at 1: PM Mon

3
4

~~308~~

1220

Jim Backlund
334 N. 14 st.
Corvallis

+ B

∠ 30° with H

45° " V

Point ½ above H

1" in front V

Paper. — Lamars.

For Willey's descrip class

# ELEMENTS

OF

# DESCRIPTIVE GEOMETRY

BY

CHARLES E. FERRIS

PROFESSOR OF MECHANICAL ENGINEERING
UNIVERSITY OF TENNESSEE

NEW YORK ·:· CINCINNATI ·:· CHICAGO
AMERICAN BOOK COMPANY

Ferris' Geom.
W. P. 10

# PREFACE.

INQUIRY among the leading draftsmen shows that nearly all their work is done in the third quadrant, or angle. It seems reasonable, therefore, to teach the subject of Descriptive Geometry in our technical schools as it will be used by our graduates. Many years of experience teaching this subject proves to the author that the student may learn to think with this problem below the horizontal and behind the vertical and perpendicular planes, just as well as above and in front of those planes.

A large number of the practical examples given in the text were assigned to cadets at the United States Naval Academy. It is believed that they will serve well to illustrate the principles of Descriptive Geometry.

C. E. F.

University of Tennessee,
Knoxville.

T-square, triangle, drawing set, etc.
P. M. Friday.

# CONTENTS.

## CHAPTER I.

### DEFINITIONS AND FIRST PRINCIPLES.



## CHAPTER II.

### PROBLEMS ON THE POINT, LINE AND PLANE.

## CHAPTER III

### SINGLE CURVED SURFACES



## CHAPTER IV

### DOUBLE CURVED SURFACES



## CHAPTER V

### INTERSECTION OF SINGLE AND DOUBLE CURVED SURFACES BY PLANES AND THE DEVELOPMENT OF SURFACES



## CHAPTER VI.

### INTERSECTION OF SOLIDS

## CHAPTER VII.

### WARPED SURFACES.



## CHAPTER VIII.

### SHADES AND SHADOWS.



## CHAPTER IX.

### PERSPECTIVE.

# DESCRIPTIVE GEOMETRY.

## CHAPTER I.

### Definitions and First Principles.

1. Draw any three planes perpendicular to each other, as the planes H, V and P, Fig. 1. From any point in space, as A, let fall perpendiculars upon these planes, piercing them in the points $A_h$, $A_v$ and $A_p$. Then the point A is fully located in space. For, if perpendiculars to the planes H, V and P be erected from the points $A_h$, $A_v$, $A_p$, they will intersect in the point A. Other points may be similarly located. Enough points will determine lines, surfaces and solids.

The operations necessary to locate a point in space from its known distances to reference planes, or to record distances on those planes, constitute the elementary principles of Descriptive Geometry.

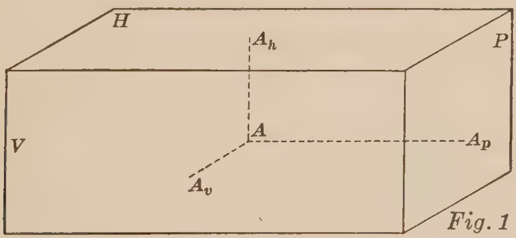


*Fig. 1*

2. Definition. Descriptive Geometry is that branch of mathematics which has for its task the determination of methods of showing by drawings the size, shape and position of objects in space.

3. The eye of the observer may be assumed to be anywhere in space. *The point in which a line from the eye, through an assumed point, pierces a plane of reference is the projection of the*

*given point on that plane.* These planes are usually the *horizontal plane*, a *vertical plane*, and a second vertical plane perpendicular to the first, and, therefore, also perpendicular to the horizontal plane, known as the *perpendicular plane.* For convenience, these reference planes, also frequently called co-ordinate planes, will be referred to as the planes H, V and P.

4. The point at which the eye is situated is the *point of sight.* When the eye of the observer is assumed to be at an infinite distance from the reference plane, the projection is *orthographic.* When the eye of the observer is at a finite distance from the reference plane, the projection is *scenographic,* or more generally the projection is called the *perspective* of the point.

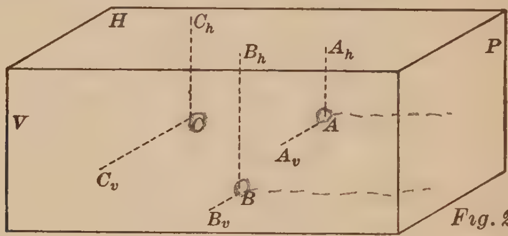


*Fig.* 2

5. The lines $AA_h$, $AA_v$, $AA_p$ are the *projecting lines* of the point A. In orthographic projection all the projecting lines, drawn to the same plane of reference, are parallel, since they are assumed to be drawn from the eye, located at an infinite distance. Thus, if we have any number of points A, B, C, etc., Fig. 2, the lines, $AA_h$, $BB_h$, $CC_h$. etc., would be parallel. So also the projecting lines upon the other planes V and P. In the scenographic projection or perspective, the lines $AA_h$, $BB_h$, $CC_h$, etc., would not be parallel, since they are all assumed to be drawn from the eye, located at a finite distance from the plane H, through the points A, B, C.

It is well for the student to fix these first conceptions clearly in mind by means of simple card-board models to represent the co-ordinate planes, and cords or wires to represent lines.

6. Assume any two points A and B, as shown in Fig. 3, and join them with a straight line. Planes through AB, perpendicular, respectively, to the reference planes H, V and P, contain the projecting lines from all points of AB and intersect H, V and P in the lines $A_hB_h$, $A_vB_v$ and $A_pB_p$, respectively. Since all perpen-

diculars from AB upon H pierce H along the line $A_hB_h$, the horizontal projection of AB is the line $A_hB_h$. Similarly the vertical projection is the line $A_vB_v$, and the perpendicular projection is the line $A_pB_p$. The plane passed through AB perpendicular to the plane H, containing the points A, $A_h$, B, $B_h$, is the *horizontal projecting plane* of the line AB. Similarly the plane through AB perpendicular to V is the *vertical projecting plane*, and the plane through AB perpendicular to P is the *perpendicular projecting plane.*

It follows from (6) that the projection of any straight line on any plane may be had by joining with a straight line the projections of any two of its points on that plane.

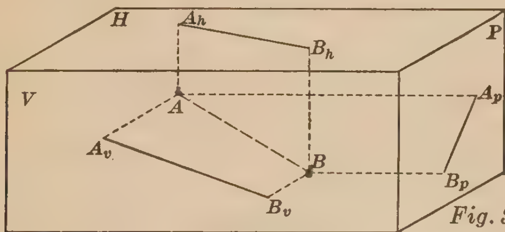



*Fig. 3*

7. The intersection of any two co-ordinate planes is known as their *ground line*. Thus, the intersection of the planes H and V is a ground line; also H and P, and V and P.

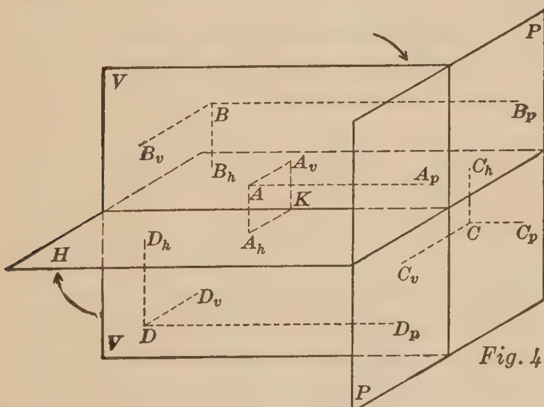



*Fig. 4*

8. The reference planes are assumed to be indefinite in extent in all directions. It follows that V extends above H and that H continues to the front of V. Since it is not always possible to keep the solution of a problem below H and behind V, it be-

comes necessary to designate the dihedral angle in which the solution may possibly fall. These dihedral angles are known as *quadrants* and are numbered consecutively, beginning with the eye of the observer assumed above H and in front of V. Thus, in Fig. 4, the angle above H and in front of V is the *first quadrant,* the angle behind V and above H is the *second quadrant*, the angle behind V and below H is the *third quadrant,* and the angle in front of V and below H is the *fourth quadrant.* For example, in the figure the point A is in the first quadrant, B in the second, C in the third and D in the fourth. In like manner, any line, as AB, in Fig. 3, may be fully located in any quadrant.

9. It is readily seen from the principles and examples given above, that no matter how an object may be located or how complex it may be, by getting the three projections of all of its lines and contours, drawings may be made, from which the object may be reproduced. Though, in general, two projections are sufficient, complex cases call for projections on all the principal planes, and in addition, projections on oblique planes.

10. To make use of the methods of projection, for the purposes of the engineer, it is necessary that *all the projections of an object be on the same plane and that the plane of the paper be this plane.* With the eye of the observer in the first quadrant, above H and in front of V, it is therefore agreed that V shall be revolved into H, backward. The perpendicular plane may be revolved either to the right or left, depending on the location of the object to be drawn and the view desired.

## Representation of Points.

11. Disregarding P, we revolve V into H as shown in Fig. 5. In the revolved position the point A, located in the first quadrant, has its vertical projection above the ground line of HV and its horizontal projection below. A point B in the second quadrant has both its vertical and horizontal projections above the ground line. A point C, in the third quadrant, has its horizontal projection above the ground line and its vertical projection below, an arrangement opposite that of the first quadrant. In the fourth quadrant both projections are below the ground line, as illustrated by D in the figure, whereas in the second quadrant both projections were above the ground line. Since these points are any

points in space, it follows that the projections of all points in space follow these laws. For example, any point in the third quadrant will have its horizontal projection above the ground line and its vertical projection below.

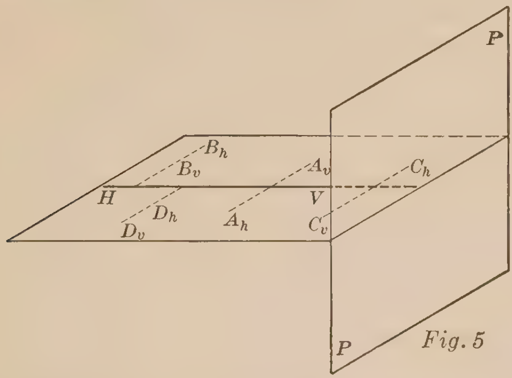



*Fig. 5*

12. In Fig. 4, since H and V are perpendicular to each other and since $AA_h$ and $AA_v$ are drawn perpendicular, respectively, to H and V, it follows that the distance from $A_v$ to the ground line equals the distance $AA_h$, or the distance of the point A above H. Also, that the distance from $A_h$ to the ground line equals $AA_v$, or the distance of the point A from V. A plane through A, $A_h$, $A_v$, cuts the line $A_vK$ from V and the line $A_hK$ from H, both being perpendicular to the ground line at K. When V is revolved into H, $A_vK$ and $A_hK$ will be one and the same straight line,

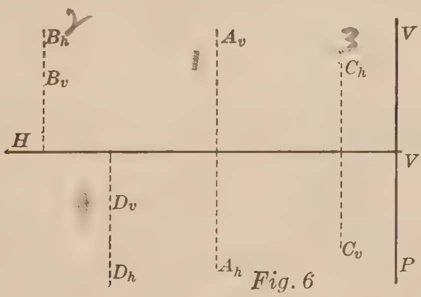



*Fig. 6*

perpendicular to the ground line at the point K. Therefore, disregarding the distance from the reference plane P, to draw the projections of a point in the third quadrant, lay off below the ground line a distance equal to the distance of the point below H. Drop a perpendicular to the ground line and produce it. Lay off on this perpendicular, above the ground line, a distance equal to the distance of the point behind V. Departing from the pictorial

and still neglecting the distance from P, we have in Fig. 6, the projections of a point C in the third quadrant 5/8″ below H and 9/16″ behind V, as shown by the usual methods of Descriptive Geometry. In a similar manner the point B is shown in the second quadrant 3/4″ behind V and 1/2″ above H; A is in the first quadrant 3/4″ above H and 3/4″ in front of V, and D is in the fourth quadrant 1/4″ below H and 3/4″ in front of V. In Fig. 6 H is in the plane of the paper, and V has been revolved about the ground line lettered VH into H. Then all points below H, without regard to quadrant, will have their vertical projections below the ground line and all points above H will have their vertical projections above the ground line. We may also show that all points behind V will have their horizontal projections above the ground line and all points in front of V will have their horizontal projections below the ground line.

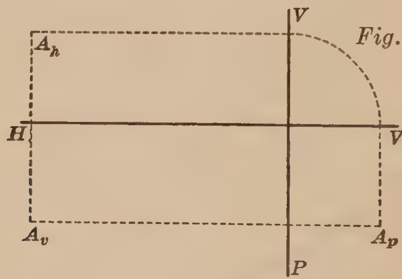


*Fig. 7*

Fig. 7 shows the point A projected on the three co-ordinate planes, A being 5/8″ below H, 9/16″ behind V and 1⅝″ to the left of P. The plane P was first revolved about the ground line PV into V, and then with V was revolved into H.

Ex. 1. Show the projections of a point 6 inches to the right of P, 4 inches in front of V, and 2 inches above H.

Ex. 2. Show the projections of a point 7 inches to the right of P, 6 inches behind V, and 3 inches below H.

Ex. 3. Locate a point by its projections—its distance from P being 2 inches, in front of V 4 inches, below H 6 inches.

Ex. 4. Show the projections of a point 5 inches to the right of P, 2 inches behind V, and 4 inches above H.

Ex. 5. Show the projections of a point 4 inches to the left of P, 3 inches in front of V, and 5 inches above H.

13. Neglecting all quadrants except the third in Fig. 4, we get Fig. 8. If in Fig. 8, A be moved into V, its horizontal projec-

tion will fall on the ground line and the point A and its vertical projection $A_v$ will coincide. Thus, in Fig. 9, the point A being in the vertical plane, its horizontal projection will be on the ground line. The point B is in the horizontal plane and has its vertical projection on the ground line. What has been shown for H and V applies to P. Therefore, *if a point be in any plane of projection, its projections on either of the other planes will be in a ground line.*

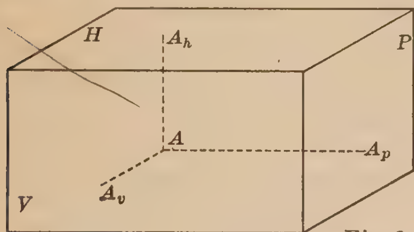



Fig. 8

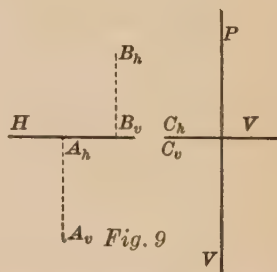



Fig. 9

This important fact is made use of in locating lines. For example, the horizontal projection of the point in which a line passes through or pierces the V plane is always found in the ground line. If a point, as the point C, of Fig. 9, be located in the ground line HV, its projections on those planes coincide and its projection on the P plane is at the intersection of the ground lines.

NOTE: It is not expected that the student will find it necessary or profitable to draw a picture of the planes of projection with the given line or lines in space and their projections as shown on Fig. 10, for example. These are given in the text with the idea of thus making the early and very important conceptions in Descriptive Geometry more clear to the beginner.

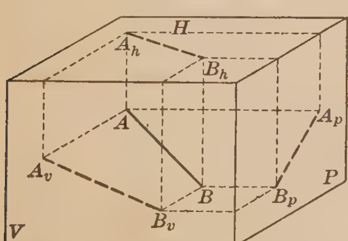



Fig. 10

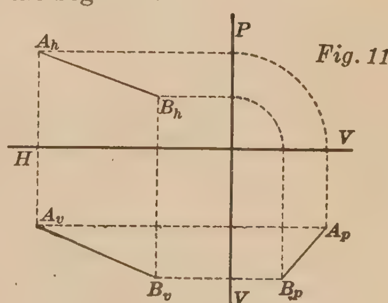



Fig. 11

## REPRESENTATION OF LINES.

14. Let AB, Fig. 10, and Fig. 11, be any line in space. Passing planes through AB perpendicular, respectively, to H, V and P, the projections are found upon these planes.

(a) If the line be located in either plane of projection it will be its own projection on that plane, and its other projections will be found in the ground lines of that plane.

(b) If the line be parallel to either plane of projection, its projection on that plane will be parallel to the line itself and its other projections will be parallel to the ground lines of that plane.

(c) If the line be perpendicular to any plane of projection, its projection on that plane will be a point and its other projections will be perpendicular to the ground lines of that plane.

(d) If the line be parallel to two planes of projection, its projections on those planes will be parallel to their ground line and its projection on the third plane will be a point.

(e) If two projections of the line are perpendicular to a ground line at a common point, the line can only be determined by its projection on the third plane. For, an infinite number of lines in space might be assumed to satisfy this condition.

(f) Two lines, drawn in two planes of projection, perpendicular to their ground line, cannot be the projections of a line in space, unless they are drawn from a common point. For, as shown in Fig. 5, any two projections of a point must be found in the same perpendicular to the ground line of the planes on which they are projected.

(g) If two straight lines be parallel in space, their projections on the same plane will be parallel. For, their projecting planes being parallel, since they are perpendicular to the same plane and contain parallel lines, their intersections with a third plane will be parallel lines.

Note: By the aid of Figures 10 and 11, and better, by use of a card-board model of the three planes of projection, the student should be able to fix clearly in mind the principles set forth in the above paragraph.

In general, *any two lines drawn at will on the paper, may be two projections of a line in space, except as limited by (f) of Section* 14.

## Representation of Planes.

15. In Fig. 12, assume an oblique plane, R. Its intersection with the plane, H, is known as its horizontal trace. Similarly, its intersections with the planes, V and P, would be its vertical and perpendicular traces. Therefore, a plane is located by its traces.

(a) When the plane R is oblique to H and V, the three planes have a common point which is in the ground line, at the intersection of the traces which R makes with H and V. In like manner the planes R, H and P, and R, V and P intersect at a common point. Therefore, *if a plane be oblique to any two planes of projection, its traces with those planes intersect their ground line in a common point.*

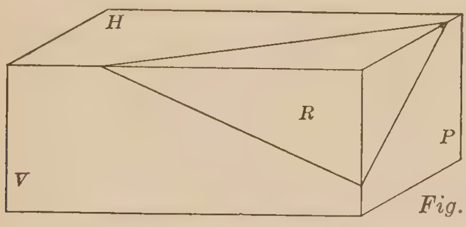


*Fig. 12*

(b) If a plane R be perpendicular to one of the planes of projection and oblique to the others, its trace with the first plane will be oblique to the ground lines of that plane, while its traces with the other planes will be perpendicular to their ground lines.

(c) If a plane R be perpendicular to two planes of projection, its traces on those planes will be perpendicular to their ground line, but parallel with the ground lines of the third plane of projection.

(d) If a plane R be perpendicular to a ground line, it will be perpendicular to the two planes of projection, which intersect in that line; for, *if a line be perpendicular to a plane, any plane containing the line will be perpendicular to that plane.* And it follows from (c), Section 15, that if the traces of a plane be perpendicular to the ground line, the plane will also be perpendicular to the ground line.

(e) If a plane R be parallel to either plane of projection, it will have no trace on that plane.

(f) If a plane R pass through the ground line of any two planes, it makes no traces with those planes other than their ground line, and can only be determined by its trace on the third plane of projection.

(g) If two planes be parallel, their traces on the same plane will be parallel; for these traces are the intersections of one of the planes of projection with the parallel planes. (See (g), Section 14.)

(h) If a plane R be parallel to a ground line but oblique to

the two planes forming the ground line, its traces with those planes will be parallel to their ground line, but oblique to the others.

Fig. 12 and Fig. 15 (a) illustrate the case of a plane oblique to all the planes of projection, as discussed in (a), Section 15. Fig. 13 and Fig. 15 (b) illustrate the case of a plane perpendicular to one plane of projection but oblique to the others. Fig. 14 and Fig. 15 (c) show the case of a plane which is parallel to one plane of projection.

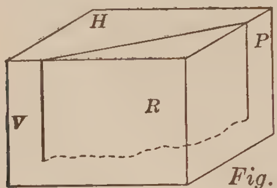



Fig. 13

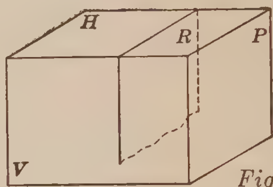



Fig. 14

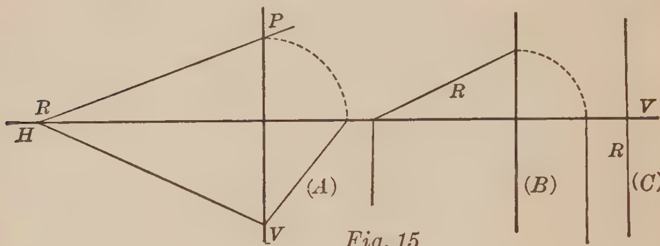



Fig. 15

## NOTATION.

16. Call the three planes of projection, horizontal, vertical and perpendicular by the letters (capitals) H, V and P. Call points by the first letters of the alphabet, beginning with A and omitting H. The projections of a point are marked by means of subscripts to denote in which plane the point is projected. For example, the point A, Fig. 16, has its projections marked $A_h$, $A_v$, $A_p$.

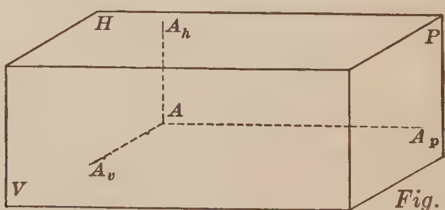



Fig. 16

Since, in the solution of a problem, two or more points of a line are known or easily assumed, it is convenient to speak of a line as the line AB, A and B being any two points of the line.

When the point or line in space ~~are~~ is considered, the subscripts are dropped, but when the projections are considered the subscripts are read as $A_h$, read "A sub h."

17. Planes may be denoted by the last letters of the alphabet, beginning with R, omitting V, and would be read, the plane R, S, etc. On the drawing, a good rule would be to put a capital letter at the intersection of the traces with a ground line. When additional letters are needed to properly designate the plane, they will be placed at the end farthest from the ground line, and subscripts used to denote which plane of projection the trace is in. If the traces are parallel, the capital letter may be placed at the left.

18. The three planes of projection give three ground lines. Call the ground lines by the planes which form them. Then the ground line between H and V will be HV, the ground line between H and P will be HP, and the ground line between V and P will be VP.

## Lines.

19. The ground line will be a heavy black line. Traces of planes, given or required, when visible, will be drawn full or solid lines. When behind other planes, other than the planes of projection, the traces will be drawn thus —— —— —— —— ——
Traces of auxiliary planes will be thus —— - - —— - - ——

Given or required lines, when visible, will be full; auxiliary and hidden lines will be broken, thus — — — — — — — — —

The projecting lines of a point will be dotted lines, thus......

A point in space, as the point A, when revolved into H, V, or P, will be marked A′. If the same point is revolved to a second position it will be marked A″, etc.

## Revolution of Objects.

20. Already has been shown the necessity of revolving V and P into H in order that the problems of the engineer may be solved in a convenient form by the graphical methods.

Let it be required to revolve any point in space into a plane of projection, about an axis in that plane. Any object is revolved about a line as an axis when the path made by each point of the object is the arc of a circle with its center on the axis, and its plane perpendicular to the axis. It therefore follows that the relative

positions of the points of the object do not change, no matter what the angle of revolution. In Fig. 17 let the point A be revolved into H about any line of H, as the line BC, as an axis. Through $A_h$ draw the line $A_hB$ perpendicular to BC, producing it indefinitely toward A′. The horizontal projection of the path of A during revolution will be on the line BA′. Join A and B. Then the triangle, $ABA_h$, is a right angle triangle with the right angle at $A_h$ ($AA_h$ being perpendicular to any line of H drawn through its foot and is therefore perpendicular to the line BA′). Revolve A

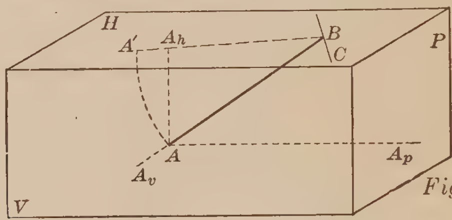



Fig. 17

to A′ about BC as an axis. Then A′ B is equal to AB and is therefore equal to the hypothenuse of a triangle of which the distance of the point A below H is the altitude and the distance of its horizontal projection from the axis of revolution is the base. But the distance of the point below H is shown by the distance of its vertical projection from the ground line HV. Therefore, to revolve a point into H about an axis in that plane, through the horizontal projection of the point draw an indefinite line perpendicular to the axis. Lay off from the axis on this line a distance equal to the hypothenuse of a triangle of which the base is the distance of the horizontal projection of the point from the axis, and the altitude is the distance of the vertical projection from the ground line. The same reasoning holds for all planes and all quadrants.

If the axis be drawn through the horizontal projection of the point, the base becomes zero and the hypothenuse of the triangle equals its altitude.

## General Principles.

21. If a line be parallel to a plane, its projection on that plane will equal the true length of the line. For, the projecting lines of the end points, the line itself and its projection, form a rectangle whose opposite sides are of equal length.

22. If a point be on a line its projections will be found on the projections of the line.

23. If two lines intersect in space, their projections will intersect, *and the projections of this point of intersection must be found on the same perpendicular to the ground line.*

24. Since a straight line may be determined by any two of its points, it may be readily determined by the points in which the line pierces any two planes of projection. These points are commonly called the traces of the line. A line may have one, two or three traces, as it is parallel to two, to one, or oblique to all the planes of projection.

25. In a similar manner, since a plane is located by any two of its lines, it may be readily located by its intersections with any two planes of projection. These lines are called the traces of the plane, and are designated as the horizontal trace, the vertical trace, or the perpendicular trace.

26. If a line be in a plane, its traces will lie in the traces of the plane.

## EXAMPLES.

1. Draw the projections of a line lying in the third and fourth quadrants. See Figs. 18 and 19. Project in each case on all the planes of projection.

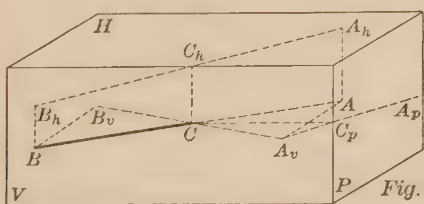


*Fig. 18*

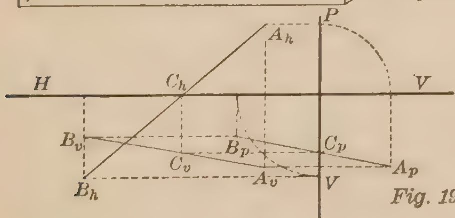


*Fig. 19*

2. Draw the projections of a line in the third quadrant, parallel to V and oblique to H.

3. Draw the projections of a line in the second quadrant oblique to V and parallel to H.

4. Draw the projections of a line in the third quadrant oblique to both H and V and passing through the ground line, HV.

5. Draw the projections of a line which intersects the ground line HV, at a point 3″ to the left of P and pierces P at a point 1 3/4″ from H and 2 1/2″ from V.

6. The point A is 2″ above H, 1″ behind V and 1/4″ to the left of P. The point B is in H, 3 7/8″ to the left of P and 1″ in front of V. Draw AB and locate the point where AB enters V.

7. Draw the projections of a line passing through the first, second and third quadrants. See Figures 20 and 21.

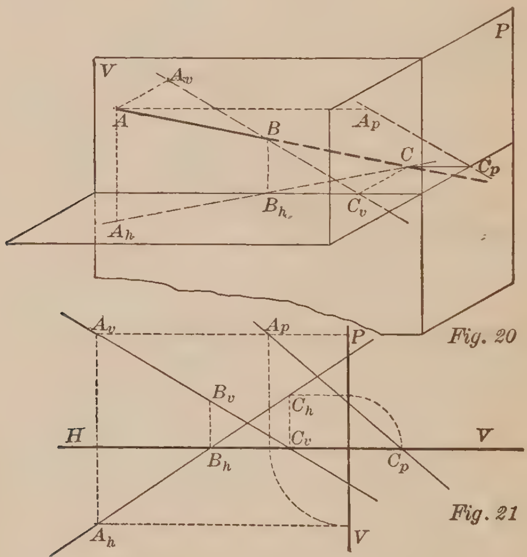



*Fig. 20*

*Fig. 21*

8. Draw the projections of a line passing through the second, third and fourth quadrants.

9. Draw the projections of a line passing through the third, fourth and first quadrants.

10. Draw the projections of a line passing through the first, second and fourth quadrants.

11. Draw the projections of a line passing through the first and third quadrants.

12. Draw a line in V below H.

13. Draw two lines intersecting in the second quadrant at a point 2″ from H and 2″ from V.

14. Draw any line through the first quadrant, through the

point A, in H and 2″ behind V. Intersect this line at the point A, with a line passing through the second, third and fourth quadrants.

15. Draw any oblique plane, as the plane R, and show its traces in all the quadrants, with all the planes of projection.

16. Draw a plane perpendicular to H but oblique to the other planes of projection.

17. Draw a plane perpendicular to V but oblique to H, showing its traces with all planes of projection.

18. Draw a plane perpendicular to P but oblique to H, and V, passing through the first and fourth quadrants.

19. Draw a plane perpendicular to P but oblique to H and V, passing through the second and fourth quadrants.

20. Draw a plane perpendicular to P and parallel to H. Also a plane similarly drawn, parallel to V.

21. In any oblique line find the point which is equally distant from H and V.

22. The point A is 3″ to the right of P, 4″ above H and 5″ in front of V. The point B is 7″ to the right of P, 3″ behind V and 4″ below H. The point C is 2″ to the right of P, 3″ behind V and 5″ above H. Draw the projections of the triangle, ABC.

23. The line AB is in the H plane. It intersects the HV ground line one inch to the right of P and makes an angle of 30° with HV. The point C is 6 inches to the right of P, 4 inches in front of V and 2 inches above H. Find the true distance from C to the line AB.

27. The student should now be familiar with the methods of projecting points and lines on the three planes of projection and should, without difficulty, work in any quadrant. Solution in the first or third quadrant will usually be easier than in other quadrants, depending on which is adopted for constant use. If any difficulty is met in other quadrants, it is well to solve the case in the quadrant where it is easiest solved, step by step, along with the solution in the required quadrant.

## Choice of Quadrant.

Nearly all text books on Descriptive Geometry teach first quadrant projection; that is, the data is so stated that the solution so far as may be is above H and in front of V. However, in many cases a part of the construction will fall in other quadrants.

For first quadrant projection assume that a cube be placed on

the H plane in front of V and to the right of P. Then, for the H projection the eye is assumed to be at an infinite distance above H, for the vertical projection the eye is assumed to be at an infinite distance in front of V, and for the P projection the eye is assumed to be at an infinite distance to the right of P. In each case the line from the eye through the point must be produced to meet the plane of projection.

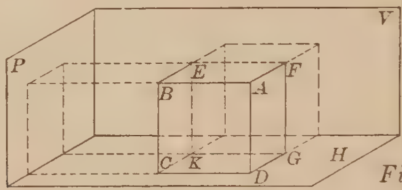



*Fig. 22*

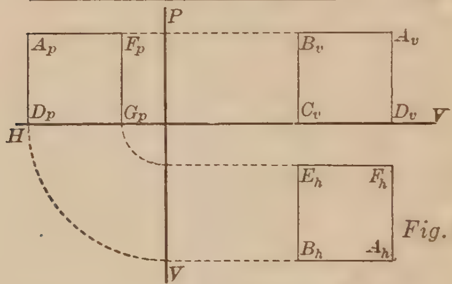



*Fig. 23*

The position of the cube in space with reference to the planes of projection is shown in Fig. 22. The projections of the cube on H, V and P planes are shown in Fig. 23, V being revolved backward into H, and P being revolved to the left into H. Considering the top of the drawing as placed in Fig. 23, then in first quadrant projection the front view of the object is at the top of the sheet, with the top view at the bottom of the sheet, and the view from the right is to the left of the sheet. This arrangement is simple enough that it need not give any trouble to a man familiar with mechanical drawing to read the drawing correctly and work from it. However, the shop man who has never studied Descriptive Geometry instinctively looks for the top view of the piece he is to build at the top of the sheet, for the front view in front of the top, for the right hand view at the right of the sheet and the left hand view to the left. This arrangement of views on the sheet is often known as the shop method of projection. It really is third quadrant, or third angle projection, with the P plane placed between the eye and the object.

A clear conception of third quadrant projection is given by considering the object to be drawn placed in a box with edges hinged. The top of the box would be a portion of the H plane behind V. The front of the box would be a portion of the V plane below H. The ends of the box would be portions of P, below H and behind V, placed at will to the right or left of the object. Fig. 24 shows the planes arranged as a box with the cube inside. The ground lines HV, HP and VP are lettered as in former figures.

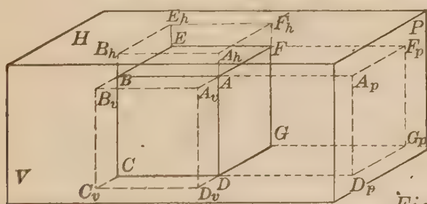



*Fig. 24*

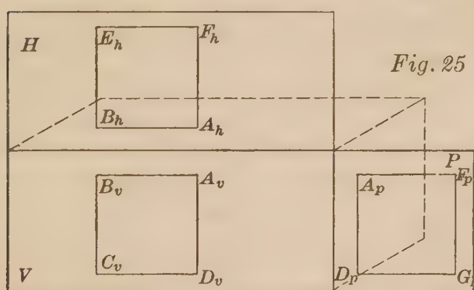



*Fig. 25*

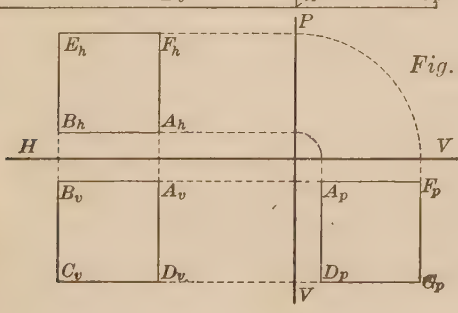



*Fig. 26*

Fig. 25 shows the top, front and end of the box revolved into a single plane, and Fig. 26 shows the projections of the cube according to the methods of Descriptive Geometry.

Inquiry among leading draftsmen in the principal designing rooms of the country shows that nearly 95% of all work is done by the third quadrant projection. It seems worth while, therefore,

to conform the class-room study of the subject to the practice which graduates must meet and adopt.

In the discussion and problems which follow, all data will be so stated as to place the solutions in the third quadrant, so far as possible. When the P plane is used, it will be placed between the eye and the object to be projected. The student will have no difficulty in getting this idea clearly in mind if he will in each case consider that the object is placed in a box, as shown in Fig. 24.

As all points, lines, planes and solids are referred to the three planes of projection, time may be saved by adopting a conventional method of determining distances from these planes and the position of points with reference to them. In order to have some fixed origin, assume the P plane to be placed at the left hand border of the sheet. As the projections on the perpendicular—also known as profile planes—are not usually needed, this assumption of P places it well out of the way, but it will be reassumed whenever needed, whether to the right or left of the object drawn. In referring to the distances of a point from the reference planes, we will give them in this order: distance from P, distance from V, H distance from H. Call measurements to right of P, in front of V, and above H, plus, and measurements to left of P, behind V, and below H, minus. The point A may therefore be located as A (6,—3,—6), being 6 inches to right of P, 3 inches behind V and 6 inches below H. It is understood that when the sign is omitted it would read plus, as for example, B(3, 4,—6), which would be read B(3,+4,—6).

## Elementary Projections in the Third Quadrant.

Ex. 1. Draw the projections of the triangle A(3,—6,—8), B(0, 0,—4), C(6,—3,—1) on all the planes of projection, showing revolved position of planes, as in Fig. 26.

Ex. 2. Draw projections of a cylinder 2″ in diameter, 4″ long. Its axis is parallel to H and V, and passes through A (1,—3,—4).

Ex. 3. Draw the line A (2,0,—4) B (6,0,—1). Find shortest distance from C (4,—6,—4) to the line.

Ex. 4. The base of a pyramid is a regular hexagon. Its center is the point (4,—4,—6). The plane of its base is parallel to the H plane. The diameter of the circumbscribed circle is 3 inches, the height of the pyramid is 4 inches. Draw its projections.

# CHAPTER II.

## 28. Problems on the Point, Line and Plane.

Problem 1. To find the traces of a given line. Let AB in Fig. 27 be the given line.

Solution. Consider first the trace of the line in V. Since the required point is in the given line its vertical projection will lie in the vertical projection of the line, and will then be a point in the vertical plane. Its horizontal projection will be in the ground line, as every point of the vertical plane will be horizontally projected in the ground line. But the point will have its horizontal projection in the horizontal projection of the given line. Therefore, the intersection of the horizontal projection with the ground line will be the horizontal projection of the vertical trace of the given line. Therefore, to find the vertical trace of a line, produce its horizontal projection to meet the ground line. A perpendicular to the ground line at this point intersects the vertical projection of the line in the required vertical trace. The same rule applies for finding the traces of a line in either H or P.

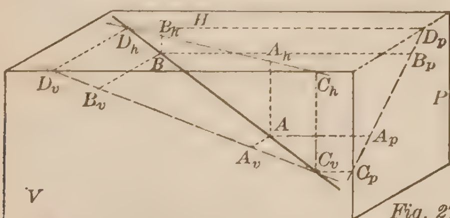



Fig. 27

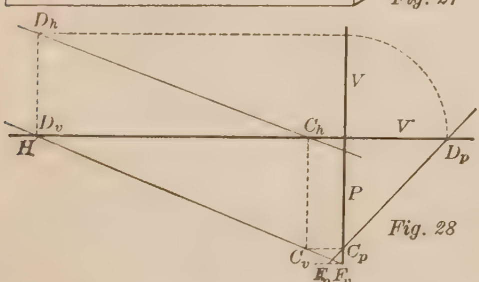



Fig. 28

Construction. Fig. 27 illustrates pictorially all the operations necessary in finding the traces, D, C and F, respectively, in the planes H, V and P. In Fig. 28, to find the trace of the line in

H, produce its vertical projection to intersect the ground line HV in $D_v$. Erect a perpendicular to HV at $D_v$. The perpendicular intersects the horizontal projection of the line, produced, in $D_h$, the required horizontal trace. Similarly, to find the trace in P, produce the vertical projection to intersect the ground line VP in the point $F_v$. A perpendicular to PV at $F_v$ intersects the perpendicular projection of the line in $F_p$, the required perpendicular trace.

Example 1. Draw the projections of a line in each quadrant and find traces in H, V and P planes, noting, particularly, how the P projection and P trace are found.

29. Problem 2. To find the true length of any oblique line in space.

In Figs. 29 and 30 let AB be the given oblique line.

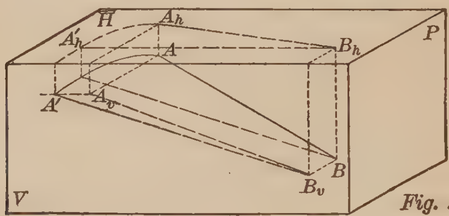



*Fig. 29*

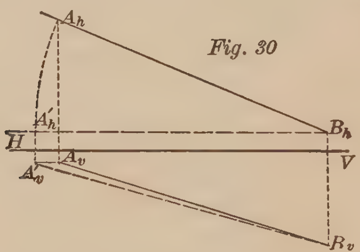



*Fig. 30*

Solution. If the horizontal projecting plane of the line be revolved about any axis perpendicular to H, until it is parallel to V, all its lines being parallel to V will be projected upon that plane in their true lengths.

Construction. With center $B_h$ and radius $A_hB_h$, strike the indefinite arc $A_hA'_h$. $B_hA'_h$ will be the horizontal projection of AB when parallel to V, and $A'_vB_v$ its vertical projection, showing the true length of the line. In like manner the line might have been revolved about an axis perpendicular to V until parallel to H and its true length projected on H. Show the true length by revolving parallel to V, also by revolving parallel to P.

30. Second Method. Fig. 31. If the horizontal projecting plane of AB be revolved about its horizontal trace into H, AB will then be a line of H and will be shown in its true length.

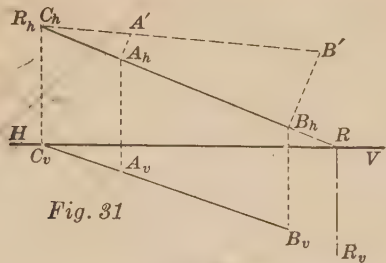



*Fig. 31*

CONSTRUCTION. Through AB pass its horizontal projecting plane R. The projection $A_hB_h$ will be in the horizontal trace of R, for AB is in the plane R and all points in R will be horizontally projected in the horizontal trace, since R is perpendicular to H. The point A will revolve in a perpendicular to the axis through $A_h$, and to a distance from $A_h$ equal to the distance of A below H. Similarly, B will revolve to B′ and $B_h$ B′ equals the distance of B below H, equals the distance of $B_v$ below the ground line. The true length of the line AB is shown by A′ B′.

By producing the line AB to meet H in $C_h$, $C_h$ being in the axis of revolution remains fixed, and if the work is correct will be in the line A′B′ produced.

The angle made by the line AB with the plane H is measured by the angle which the line makes with its projection on that plane. This angle is shown in its true size when AB is revolved into H at A′B′ in the angle $B'C_hB_h$.

31. PROBLEM 3. The traces of a line being given to determine the line. In Fig. 32, let $B_v$ be the vertical trace of the line and $A_h$ its horizontal trace.

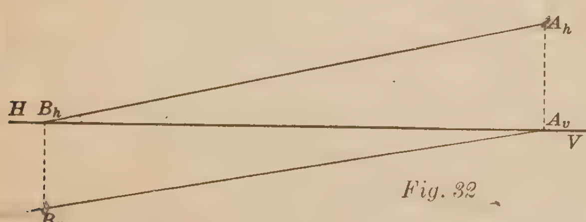



*Fig. 32*

SOLUTION. Since $B_v$ is the point of the line in V, its horizontal projection will be found in the ground line HV; and since $A_h$

is the point of the line in H, its vertical projection is in the ground line HV. Therefore, the projections of two of its points being known, the line is determined and its projections may be drawn.

CONSTRUCTION. Perpendiculars upon the ground line HV, from $A_h$ and $B_v$ locate $A_v$ and $B_h$. Joining $A_hB_h$ and $A_vB_v$ we have the projections of the line AB.

Example 1. The traces of a line are A (0,—3,—6), B (4,0—2). Draw the projections of the line AB on H, P and V.

32. Before going farther, the student should become familiar with projections of lines and points in all the quadrants and upon all the planes of projection. For example, the problems 1, 2 and 3 should be solved as readily in the first, second or fourth quadrants as in the third, or projections be found as readily on the perpendicular plane as on the vertical.

EXAMPLES.

1. Draw the projections on all the planes, of a line passing through A (2″,—3″,—4) and B (—2,—3,—1). Show the true distance from A to B. Show the angle made by AB with H.

2. Join A(3, 0,—2) and B(0,—4,—4). Find the angle made by AB with the plane P.

3. The points A(1/2, 3, 1/2) and B(3,—1, 4) determine the line AB. Find the true length lying in the first quadrant.

4. Find the true distance from A(—2″,—2″,—2″), to B (3″,—4″,—5″). Locate the point in which AB pierces H, V and P. Find the angle which AB makes with V.

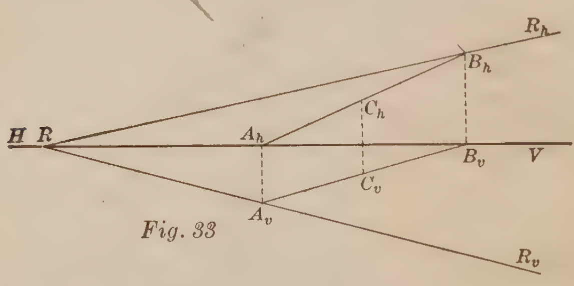


*Fig. 33*

33. PROBLEM 4. To assume a line in a given oblique plane. Let R in Fig. 33 be the given plane.

SOLUTION. Since the traces of the plane are lines in H and V, any point in the horizontal trace may be assumed to be the horizontal trace of some line of the plane. In like manner any point

of the vertical trace may be assumed to be the vertical trace of some line of the plane. Therefore, since two points of a line may be determined, the line is known.

CONSTRUCTION. Assume any point in the H trace, as $B_h$. Its vertical projection will be in the ground line HV, as every point in it is vertically projected in the ground line. Similarly, $A_v$ is any point in the vertical trace of the plane. Its horizontal projection will be in the ground line HV. Draw the projections of AB through the projections of the points A and B.

Solve the same problem, using the P plane in place of the H plane.

34. Second Method. Fig. 33.

SOLUTION. Since an infinite number of lines may be horizontally projected in the line $A_hB_h$, we may assume one to be the projection of a line of the given oblique plane. Its intersection with the ground line HV will be the horizontal projection of the vertical trace of the line. Its intersection with the horizontal trace will be the horizontal trace of the line. Therefore, the projections of two points of a line being known, the line is determined.

35. To assume a point in an oblique plane it is only necessary to assume one projection. Through this projection pass any line of the plane. The other projections of the point will be found in the other projections of the line.

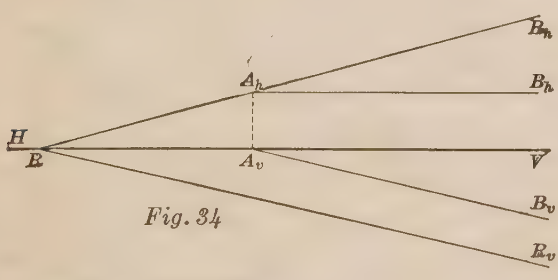



Fig. 34

36. If a line of a plane be drawn parallel to one of its traces, its other projections will be parallel to the ground lines. For, let the line AB in Fig. 34 be drawn parallel to the vertical trace of the plane R. Then, the lines being drawn parallel, their projections on any plane will be parallel. But the horizontal projection of the vertical trace is in the ground line HV. The horizontal projection of AB will therefore be parallel to the ground line HV.

The same reasoning applies to projections on the perpendicular plane.

NOTE: Fig. 34 illustrates the most convenient auxiliary line of a plane, its projections being parallel to known lines of the plane. This construction is much used later in the book.

Example 1. Show the traces of a plane in the second quadrant and locate a point in the plane.

Example 2. The traces of a plane pass through the points A (4,0,0), B (6,0,—6), C (4,—5,0). Locate a point D (x,—2,—3) in the plane.

CONSTRUCTION. Draw a line of the plane at a distance of 3 inches below the H plane. Such a line will have its V projection parallel to the ground line HV and its H projection parallel to the H trace of the plane R. This line will contain the point D. Similarly, draw a line of the plane R at a distance of 2 inches behind the V plane. This will have its H projection parallel to the ground line, its V projection parallel to the V trace and will also contain the point D. The intersections of these lines will be the projections of D.

Example 3. The plane R is in the fourth quadrant. Locate in R a point one inch below H and one inch in front of V.

37. PROBLEM 5. To pass a plane through two lines which intersect, or are parallel.

The lines are AB and CD intersecting in the point E, as in Fig. 35.

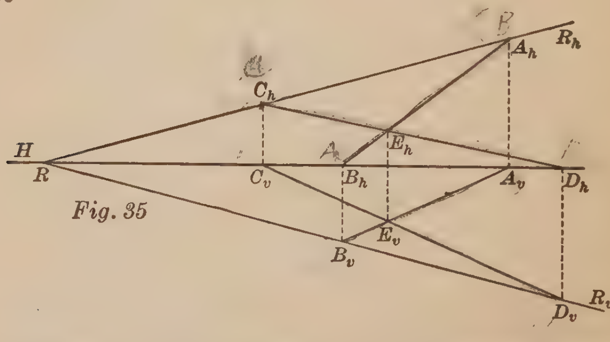


*Fig. 35*

SOLUTION. Since the lines lie in the required plane, their traces are in the traces of the plane; and since two points are sufficient to locate a line, the vertical trace is determined by the vertical traces of the two lines. In a similar manner the horizontal trace is determined.

CONSTRUCTION. Produce the projections of AB and CD to intersect the ground line to find their traces in H and V. Through the traces of the lines draw the traces of the plane R. If the work is correct, the traces of the plane will intersect the ground line at a common point. If one trace of a plane is known, one point in the other trace will be sufficient to draw the trace; for, it will either intersect the ground line at a point common to both traces, or it is parallel to the ground line. It may be necessary to produce the traces far beyond the limits of the drawing to find this common point of the traces.

38. To pass a plane through three points it is only necessary to draw lines through the points which either intersect or are parallel, and find their plane, as in Problem 5.

39. To pass a plane through a point and a line, either draw a line through the point intersecting the given line, or parallel with it.

40. If the given lines do not intersect the ground line within the limits of the drawing, new lines intersecting the given lines may be assumed which, passing through two points of the plane, lie in it and their traces, therefore, are points in the traces of the plane.

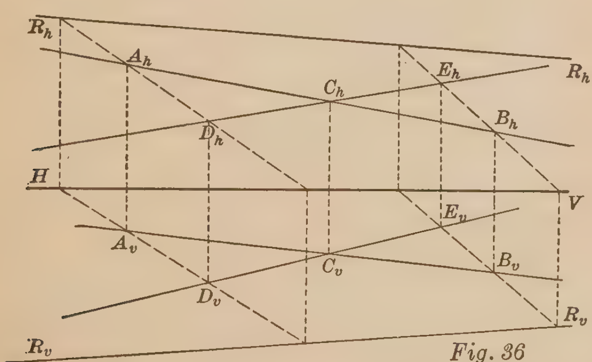



*Fig. 36*

In Fig. 36 the projections of the lines AB and CD do not intersect the ground line within the limits of the drawing. Any other lines, as AD and BE, may be assumed, intersecting the given lines and intersecting the ground line, as shown, locating the traces of the plane R.

The student should solve Problem 5 with all its special cases on the perpendicular plane.

41. Problem 6. To find the angle between two straight lines which intersect.

In Fig. 37 let AB and CD be the two given lines intersecting in the point E. To find the angle A E C.

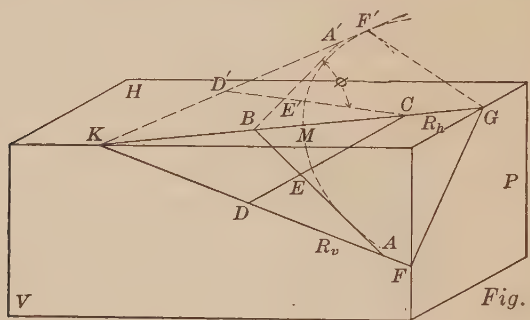



Fig. 37

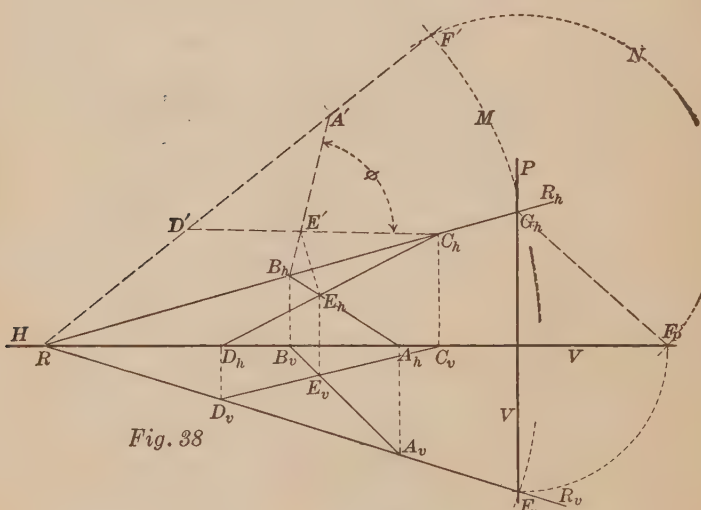



Fig. 38

Solution. If a plane be passed through the two lines and then revolved about a trace into one of the planes of projection, the true angle made by the lines in space will be shown.

Construction. The plane R is passed through the lines AB and CD, as shown in Figs. 37 and 38, by the methods of Problem 5. The planes of projection cut from the plane R a triangle, of which the vertical trace, the horizontal trace and the perpendicular trace are sides. As the lines AB and CD are drawn in this triangle,

it is only necessary to revolve it into H to show the angle which they make with each other. Let the triangle be revolved about its horizontal trace as an axis. Then F, in Fig. 37, will travel in the path F M F′ to F′, the vertical trace will be revolved to K F′ and the perpendicular trace to G F′. The line CD will revolve to C D′ and AB to A′B. But with the triangle revolved into H, the true value of the angle $\Phi$ is shown.

In Fig. 38 the revolution is shown by the methods of Descriptive Geometry. $F_vR$ being a line of the V plane, is projected in its true length. With center R and radius $RF_v$ strike the indefinite arc $F_v$ M F′. The side of the triangle in P is shown in its true length in the P trace of the plane R. However, by making use of the P plane as a co-ordinate plane and revolving it into H about HP, the true length is shown in $F_1G_h$. With center $G_h$, and radius $G_h$ $F_p$, strike the indefinite arc $F_p$ N F′. The revolved position of the triangle is R $G_h$ F′, the point D is found at D′, the length R D′ being made equal to the length $RD_v$. The point A is found at A′. The required angle is shown at $\Phi$.

42. It readily follows from Problem 6, that all angles and lines of the triangle G R F are shown in their full sizes when the plane is revolved into H. Therefore, the angle $G_h$ R F′, is the angle made by the vertical and horizontal traces in space. Similarly, R F′ $G_h$ is the angle between V and P traces, and R $G_h$F′ is the angle between the P and H traces.

43. Problem 7. To bisect the angle formed by two right lines in space and show the projections of the bisector.

Let the lines be given as in Fig. 38. To bisect the angle A E C.

Solution. If the plane of the lines forming the angle be revolved into one of the planes of projection, the angle will show in its true size and its bisector may be drawn. If the plane be revolved to its original position and the projections of one point of the bisector be located, the projections of the vertex of the angle being known, the projections of the bisector may be drawn.

Not only may the bisector of the angle be located, but the plane being in one of the planes of projection, all its lines, angles and points show in their true relative positions.

Example 1. A plane R passes through the points A (6,—3,—2), B (2,—8,—5), C (8,—4,—8). Find the true shape

of the triangle A B C. Find the length of bisector of angle at A drawn to opposite side.

44. PROBLEM 8. To find the shortest distance from a point to a line.

In Fig. 39 let A be the point and BC the line.

SOLUTION. If a plane be passed through the point and the line and then be revolved about one of its traces into a plane of projection, the true relative positions of the point and the line may be shown and the projections of the shortest line from the point and the line drawn.

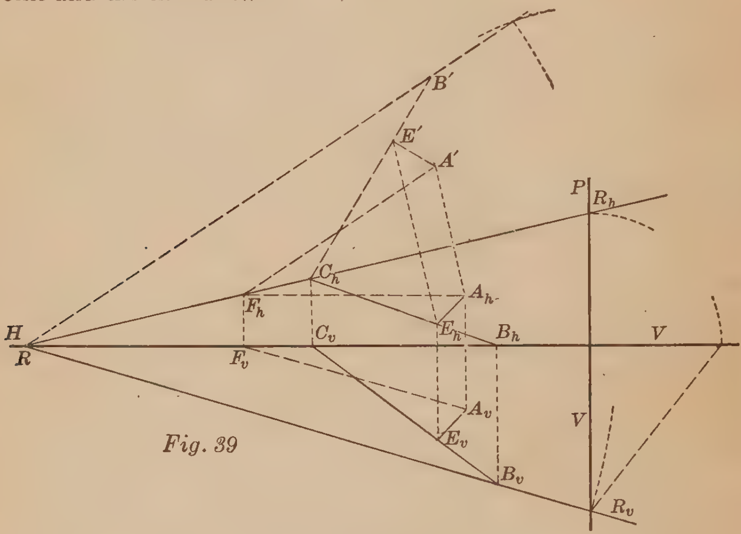


*Fig. 39*

CONSTRUCTION. Through the line BC, and the point A, pass the plane R (construction not shown). Revolve R about its horizontal trace into the H plane. The line BC revolves to $C_hB'$. To find the revolved position of the point A, through A draw any convenient line of the plane, as the line AF, and find its revolved position $F_hA'$. But the path of A in revolution will be in a perpendicular to the axis of revolution, through its horizontal projection; therefore at the intersection A'. Draw A'E' perpendicular to $C_hB'$, this being the required distance. In counter revolution the projections of AE are found. The problem should also be solved on the P plane.

45. PROBLEM 9. To find the line of intersection of two planes.

In Fig. 40 let R and S be the two planes. To find their line of intersection.

SOLUTION. If a line of one of the planes intersect a line of the other plane, a point common to both planes, and therefore a point in their line of intersection, becomes known. But the vertical and horizontal traces intersect, thus giving two points in the line of intersection. The line drawn through these points is the required line.

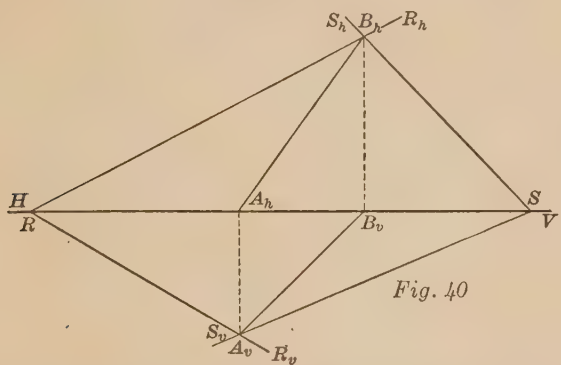



*Fig. 40*

CONSTRUCTION. The vertical traces intersect in the point $A_v$, its horizontal projection being in the ground line. The horizontal traces of the planes intersect in $B_h$, its vertical projection being in the ground line. The line of intersection is the line AB.

46. If either the horizontal or vertical traces fail to intersect within the limits of the drawing, as in Fig. 41, an auxiliary

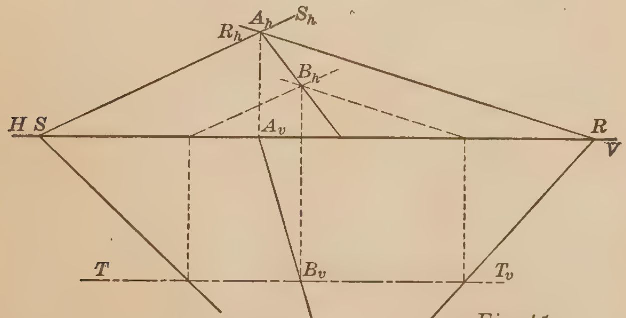



*Fig. 41*

plane may be passed parallel to H or V, intersecting the traces of the given planes. Such a plane, if parallel to H, will cut a line from each of the given planes, which will be horizontally projected parallel to the traces of the planes from which they are

cut. In the figure the vertical projections of the lines will be found in the trace of the plane T, as it is parallel to the horizontal plane and therefore perpendicular to the vertical plane. Similarly, if an auxiliary plane be passed parallel to V, the vertical projections of the lines cut from the planes will be parallel to the vertical traces of the planes from which they were cut. In Fig. 41, the plane T might have been passed parallel to V as well as to H.

47. If both vertical and horizontal traces of the planes fail to intersect within the limits of the drawing, two auxiliary planes, locating two points common to the two given planes, will determine their line of intersection.

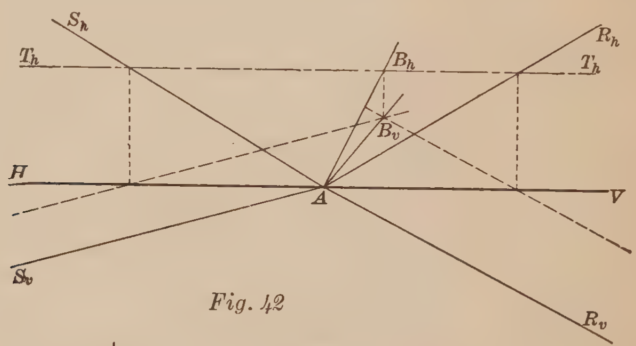



*Fig. 42*

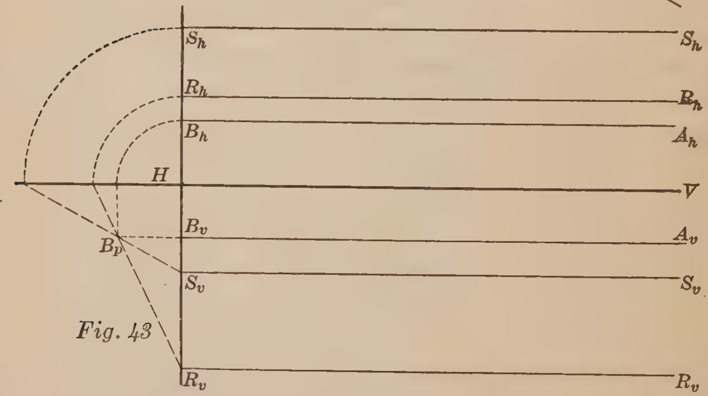



*Fig. 43*

48. If the traces of the given planes intersect the ground line at a common point, as in Fig. 42, this point will be in the required line of intersection. An auxiliary plane, parallel to H or V, intersecting the traces, will determine a second point in the line of intersection. In Fig. 42, AB is the line of intersection of the planes R and S. The point A being in the ground line, will be its own vertical and horizontal projections.

49. If the planes be parallel to the ground line HV, but oblique to H and V, as in Fig. 43, the line of intersection will be parallel to the ground line, and can only be determined by means of the third plane of projection P. In the figure, the traces of planes R and S are first found in the P plane. The P traces intersect in the point $B_p$. Since the planes are perpendicular to P, the horizontal and vertical projections of the line of intersection will be parallel to the ground line HV. The line of intersection is AB.

50. Problem 10. To find the point in which a given line pierces a given plane.

In Fig. 44 it is required to find the point in which the line AB pierces the plane R.

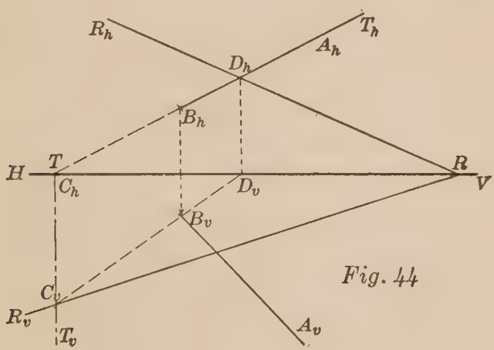


*Fig. 44*

Solution. If a plane be so passed through the given line that it will intersect the given plane, the line of intersection of these planes contains all points common to the two planes, and therefore the point in which the given line pierces the given plane.

Construction. As one line will not locate a plane, an infinite number of planes may be passed through the line AB. But the simplest one to use as an auxiliary plane in the solution of this problem is the plane perpendicular to H. Pass the plane T through AB, perpendicular to H. Then the V trace is perpendicular to the ground line HV, and the horizontal trace coincides with the H projection of the line. The line of intersection of the two planes is DC. The H projection of the given line AB, and of the line of the intersection DC, fall on the H trace of the plane T. But their vertical projections intersect in the point $B_v$. Therefore the line AB pierces the plane R at the point B.

The vertical projecting plane of the line AB could have been used as an auxiliary plane with the same results and with the same effort. The same may be said of the perpendicular projecting plane.

Example 1. Given the points A(3,—4,—5), B(4,—6,—1), C(1,—2,—3), D(6,—1,—1), E(3,—5,—6). Find the plane of ABC and find where the line DE pierces this plane. Revolve the plane about one of its traces to show the true size and shape of the triangle ABC.

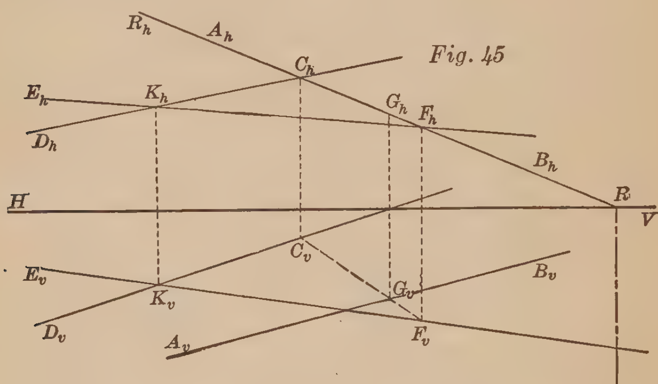


*Fig. 45*

51. If the plane be given by two of its straight lines, the order may be reversed and the points in which these lines pierce the horizontal projecting plane of the given line be found. In Fig. 45 let it be required to find the point in which the line AB pierces a plane given by two of its lines CD and EF. Pass through AB its horizontal projecting plane R. The line CD pierces R in the point C. The line EF pierces R in the point F. But C and F are two points in the line of intersection of the auxiliary plane R, with the plane of the two given lines. The line CF intersects AB in the point G, the required point.

52. A plane perpendicular to one plane of projection and oblique to the others is a convenient auxiliary plane. Usually, this is assumed perpendicular to H and oblique to V and P. One of the uses of such a plane is to locate a point in an oblique plane, when one of its projections is assumed or given. In Fig. 46 assume the H projection of a point at $A_h$. Through $A_h$ draw a line perpendicular to H, vertically projected in $A_vB_v$. The vertical projection of the point in which this line pierces the plane will be the vertical projection of the required point of the plane.

This is found, as in Problem 10, by use of the auxiliary plane perpendicular to H. The vertical projection of A is the vertical projection of a point in the plane.

It will be seen that the plane T cuts a line from the plane R and that the projections of the line contain the projections of the point. Therefore we may, through the assumed projection of a point of the plane R, draw the projection of a line. The other projections of the line will contain the other projections of the point.

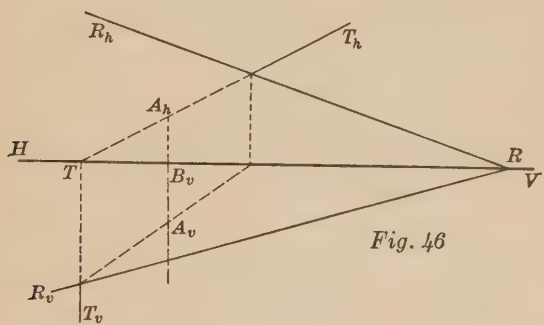



*Fig. 46*

Another convenient auxiliary plane is perpendicular to H but has its H trace parallel to the H trace of the given plane. If this be used to find the vertical projection of a point in a plane, when the horizontal projection is assumed, pass the plane through $A_h$, parallel to $RR_h$. This line cuts the line AB from R and the vertical projection of A is at $A_v$. Or, briefly, draw the horizontal projection of a line through $A_h$, parallel to the H trace of the plane R. If AB be in the plane R, its vertical projection will be parallel to the ground line HV, and the vertical projection of A is found at $A_v$. Construction is shown in Fig. 47.

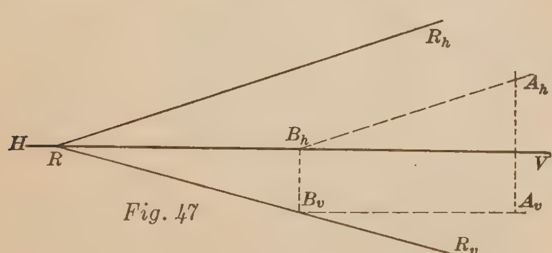



*Fig. 47*

53. If a straight line be perpendicular to a plane, its projections will be perpendicular, respectively, to the traces of the plane.

The horizontal projecting plane of the line, since it contains a line perpendicular to the given plane, is perpendicular to that plane. But, by construction, it is perpendicular to H. Therefore the horizontal projecting plane of the line is perpendicular to the horizontal trace of the given plane, being perpendicular to the two planes intersecting in that line. All lines of the horizontal projecting plane have their horizontal projections in its horizontal trace. Then the horizontal projection of the given line is in this trace and is perpendicular to the trace of the given plane.

It follows, therefore, that to assume a straight line perpendicular to a plane, it is only necessary to draw its projections perpendicular, respectively, to the traces of the plane. Or, to assume any plane which shall be perpendicular to a line, through any point in the ground line draw the traces perpendicular, respectively, to the projections of the line.

54. PROBLEM II. From a given point to draw a line perpendicular to a given plane and to find the distance from the point to the plane.

In Fig. 48 it is required to find the distance from the point A to the plane R.

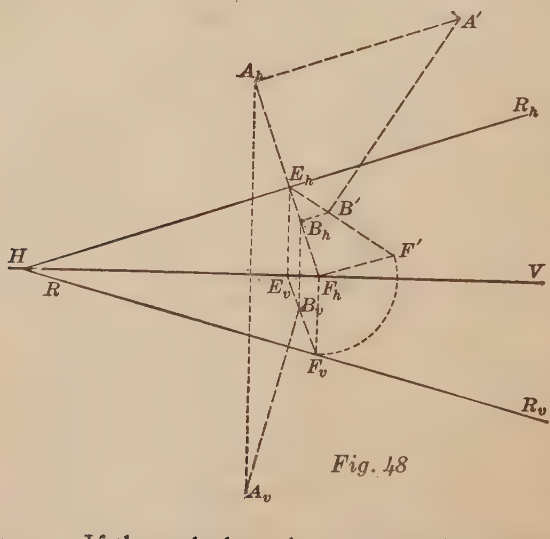


*Fig. 48*

SOLUTION. If through the point a perpendicular be drawn to the plane and the point be found in which it pierces the plane, the distance from this point to the given point will be the required distance.

CONSTRUCTION. Draw through the point A the line AB, perpendicular to the plane R, and piercing it at B. Revolving the horizontal projecting plane of AB about its horizontal trace into H, the true distance AB is shown in the line A′B′.

The problem should be solved in other quadrants than the third, and also when projected in P.

55. PROBLEM 12. To project a given right line on a given plane and show the true position of the projection with reference to any other line of the plane.

In Fig. 49 let AB be the given line and R the given plane. To project AB upon R and find where this projection intersects CD which is any line of the plane R.

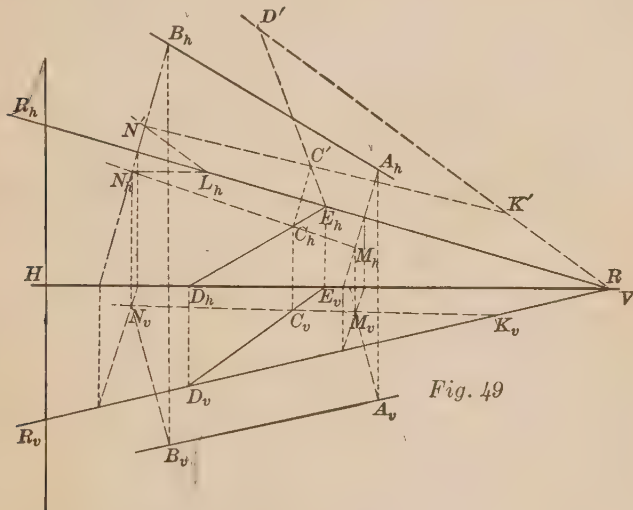



*Fig. 49*

SOLUTION. If from any two points of the line, perpendiculars be drawn to the plane, they will pierce it in points of the required projection. By revolving the plane about one of its traces into one of the planes of projection, all the lines of the plane show their true relative positions.

CONSTRUCTION. Perpendiculars from the points A and B pierce R, in the points M, N, and the projection of AB on the plane R is in the line MN. Assume any line of the plane, as the line DE, intersecting the line MN, in the point C. To find the true relative position of these lines, R is revolved into H about the H trace, or into V or P about their respective traces. The line DE, intersecting both traces, is readily found in its revolved

position. In the figure, MN intersects the vertical trace at $K_v$, revolved to K′. To find another point of MN, an auxiliary line is drawn through any point as $N_h$, parallel to V, therefore parallel to the vertical trace of R, and its revolved position will be parallel to the revolved position of the vertical trace. The point L remains fixed, being in the axis, and $N_h$ revolves to N′. The line DE, and the line MN, show in their true relative position at $E_hD'$ and K′N′.

56. Example 1. The points A(0, 0,—2), B(2, 0,—4) and C(0,—2,—7) determine a plane R. Project the line, E(3,—2,—3), F(2,—2,—6) upon the plane R and find the angle made by the projection with the line AB.

57. Example 2. Four points are located in space; A(3,—2, 4), B(2,—2,1), C(2,—5,—7) and D(4, 5,—3). Find the plane of ABC and find how far D is from their plane.

58. Example 3. The points A(2, 0,—4), B(0, 0,—2), and C(0,—2, 7) locate a plane. Find where a line passing through the points D(2,—1 1/2, 0) and E(4, 1 1/2,—4) pierces this plane. Project the line on the plane and find the angle which the projection makes with the horizontal trace of the plane.

59. Problem 13. To find the angle which a given line makes with an oblique plane.

In Fig. 50 it is required to find the angle which the line AB makes with the plane R.

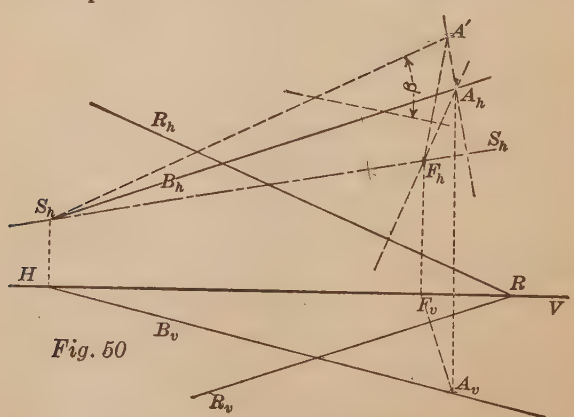



*Fig. 50*

Solution. Produce the line to meet the plane. Then the line in space, its projection on the given plane and a perpendicular

upon the plane from any point of the line, form a right triangle, in which the angle made by the line with its projection on the plane is the required angle. If the plane of the triangle be revolved to a position parallel to either plane of projection, the true value of this angle will be shown. But since the triangle contains a right angle, the angle made by the given line with the perpendicular is the complement of the angle made by the given line with its projection. Therefore, if a plane be passed through the given line and a perpendicular from any point of that line to the given plane, and this be revolved about either trace into a plane of projection, the angle which the lines make will be the complement of the required angle.

CONSTRUCTION. Find the horizontal trace of the plane passed through AB and AF, (AF being perpendicular to the plane R). Revolve this plane about its horizontal trace into H. Any line perpendicular to A′F will be parallel to the projection of AB on R and will make with AB the required angle, shown at B in the figure.

60. PROBLEM 14. Through a given point to pass a plane perpendicular to a given right line.

In Fig. 51 it is required to pass a plane through the point A, perpendicular to the line BC.

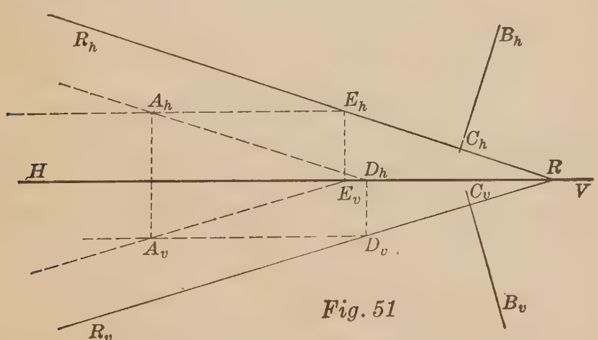


*Fig. 51*

SOLUTION. Since the plane is to be perpendicular to the line its traces will be perpendicular, respectively, to the projections of the line. Therefore, if through the given point a line be drawn whose horizontal projection is perpendicular to the horizontal projection of the given line, it will be a line of the required plane, parallel to its horizontal trace. Since the direction of the trace is known, one point is sufficient to draw it.

Construction. A line through A perpendicular to the horizontal projection of BC, being parallel to the horizontal trace of the required plane, is parallel to H, and its vertical projection is parallel to the ground line. Through the vertical trace of the line draw the vertical trace of the plane perpendicular to the vertical projection of the given line. A similar construction will determine the horizontal trace; or, at the intersection of the vertical trace with the ground line, draw the horizontal trace, perpendicular to the horizontal projection of the given line.

61. Problem 15. To pass a plane through a given right line parallel to another right line.

In Fig. 52 it is required to pass a plane through AB which will be parallel to the line CD.

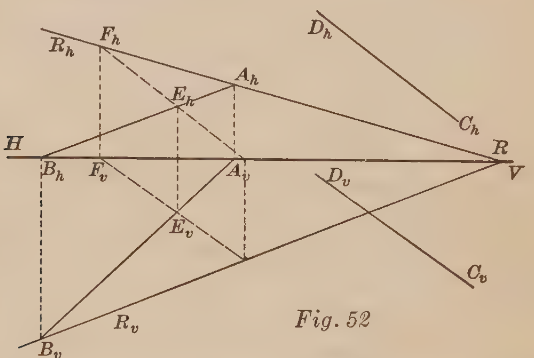



*Fig. 52*

Solution. Through any point of the given line draw a line parallel to the second line. A plane through these intersecting lines will be the required plane.

Construction. Through any point of AB as E, draw the line EF, parallel to CD.

62. If it be required to pass a plane through a point, parallel to two given right lines, it is only necessary to draw through the point two auxiliary lines, parallel, respectively, to the given lines, and through these intersecting lines pass a plane.

63. Let it be required to pass a plane through a given point parallel to a given plane. Since the planes are parallel their traces are parallel, and one point in each trace will be sufficient to determine the plane. Therefore, draw any convenient line of the given plane and through the given point draw a parallel line,

which will be a line of the required plane. Through the traces of this line draw the traces of the plane.

64. PROBLEM 16. To pass a plane through a given line perpendicular to a given plane.

SOLUTION. If one line of a plane be perpendicular to a second plane, the planes are perpendicular. Therefore, through any point of the given line draw a line perpendicular to the given plane. A plane passed through these two lines will be the required plane.

65. PROBLEM 17. To find the angle between two given planes.

In Fig. 53 it is required to find the angle which the plane R makes with the plane S.

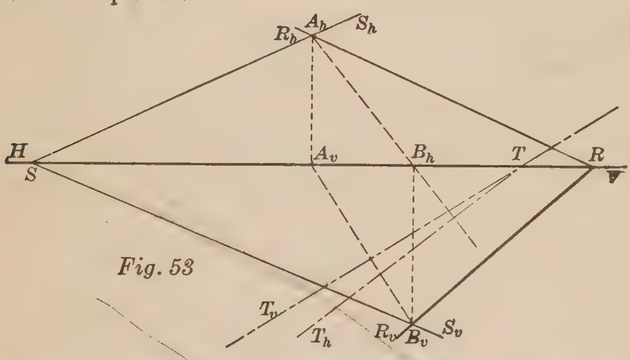



*Fig. 53*

SOLUTION. If a line be drawn in each plane from the same point in their line of intersection and perpendicular to it, the angle which these lines make with each other is the measure of the plane or dihedral angle. Therefore if a plane be passed perpendicular to the line of intersection of the two planes (therefore, perpendicular to both planes) it will cut from each a line, which make with each other the angle which the planes make in space.

CONSTRUCTION. The planes R and S intersect in the line AB. The plane T is perpendicular to the two planes, being perpendicular to their common line AB. Find the lines of intersection of the plane R with T and of S with T. These lines of intersection make with each other the required angle, which may be found by revolving the plane T containing the angle, into one of the planes of projection. All the construction is omitted in Fig. 53, as the mass of lines would make it difficult to understand the statement of the problem. The student should find no difficulty in constructing the figure.

66. Second Method. PROBLEM 17. Fig. 54.

SOLUTION. If from any point in space a perpendicular be dropped to each of the given planes and a plane be passed through these perpendiculars, it will cut from each of the planes a line. The perpendiculars are at right angles, respectively, with the lines cut from the planes, making with each other the supplement of the required angle. Therefore, if from any point in space perpendiculars be dropped to the planes, the angle between the perpendiculars will be the supplement of the required angle.

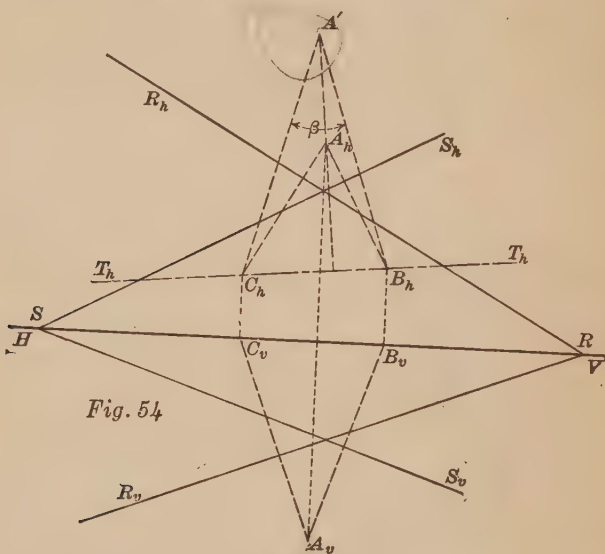



*Fig. 54*

CONSTRUCTION. From the point A, the line AB is drawn perpendicular to S and AC is perpendicular to R. Pass through these perpendiculars the plane T, (only its H trace is needed) and revolve it into H. The point A falls at A′. The angle B is the supplement of the angle which the planes make in space.

67. PROBLEM 18. To find the angle which any plane makes with any plane of projection.

In Fig. 55 it is required to find the angle which the plane R makes with the plane H, V, or P.

SOLUTION. Since the measure of a plane or dihedral angle is the angle made by a line in each plane, drawn from a point in their line of intersection and perpendicular to it, we have only to pass a plane which is perpendicular to the given plane and one of the planes of projection. This perpendicular plane will cut

from each a line. These make with each other the required angle. By revolving into one of the planes of projection, the true value of the angle is shown.

CONSTRUCTION. Pass an auxiliary plane perpendicular to the horizontal trace of the plane R, through $A_hB_h$. This makes the auxiliary plane perpendicular to both H and R, being perpendicular to their common line. Revolved about its H trace into H, B falls at B′. The line AB, a line of R, falls at $A_hB'$. B is the angle.

Problem 18 is only a special case of Problem 17, when one of the given planes is a plane of projection. The line AB is cut from R and the horizontal projection of AB, $A_hB_h$ is cut from H. The angle B between these lines is the required angle.

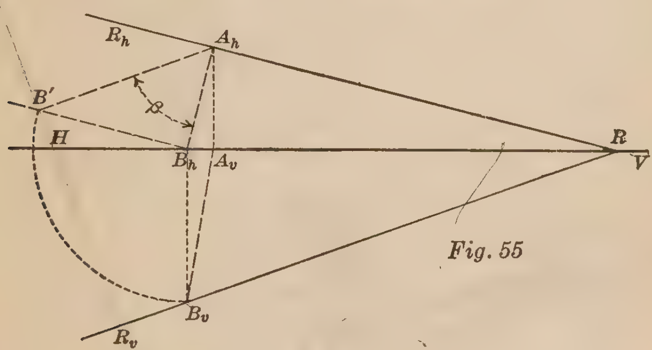



Fig. 55

68. PROBLEM 19. One trace of a plane (as the horizontal) being given and the angle which the plane makes with the corresponding plane of projection, to find the other trace.

In Fig. 55 the trace $RR_h$, and the angle B, being given to find the trace $RR_v$.

69. EXAMPLE. The H trace of a plane passes through the points A(0, 0, 0) and B(3,—4, 0), the plane makes an angle of 60° with H, find the trace in V. Also find the trace in P, the angle which the plane makes with V and with P. Also find the angle which the V and H traces make with each other in space.

70. PROBLEM 20. To find the shortest line which can be drawn terminating in two right lines not in the same plane.

In Fig. 56 it is required to locate the shortest line which can be drawn from AB to CD, and to find its true length.

SOLUTION. If through one of the lines a plane be passed par-

allel to the other, and the second line be projected upon this plane, the projection will intersect the first line. If from this point of intersection a perpendicular be erected to the plane, it will intersect the second line and will therefore be the shortest line, for it will be in the projecting plane of the second line, upon the plane of the first.

CONSTRUCTION. Pass the plane R through the line AB, parallel to CD. Project the line CD upon this plane in the line MN. Since the plane R is by construction parallel to the line CD, its projection on R will be parallel to itself. The perpendicular to R, drawn from the intersection of AB with MN, intersects CD in the point C. Then CM is the required distance. Revolved into H its true length is shown in C′M′.

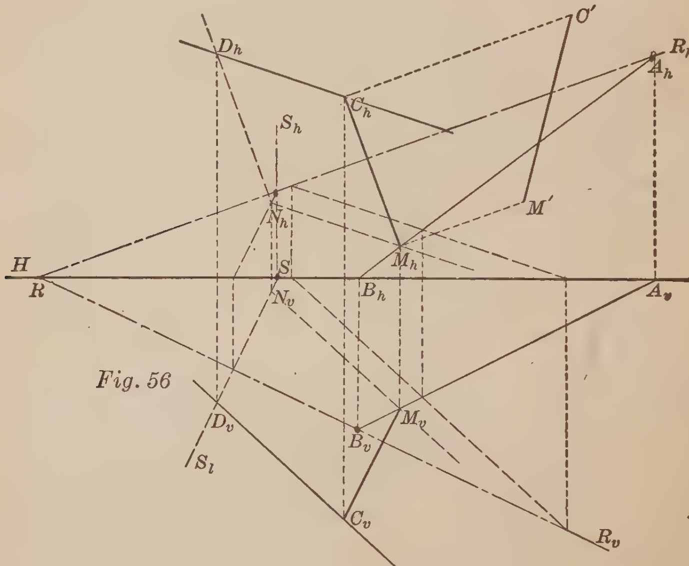


*Fig. 56*

71. PROBLEM 21. To draw the projections of a solid of definite size, occupying a fixed position in space, with its base resting on an oblique plane.

No new principles are involved in the solution of this problem. In Fig. 57 let the plane R be given by the angle which the horizontal trace makes with the ground line HV, and by the angle which it makes with H. Assume the point G, of the base 4″ below H and 8″ behind V. Assume the edge KG, making an angle of 30° with the H trace of R.

After locating G in the plane, revolve it into H about the

trace $RR_h$, and draw the base in its true size and position, as shown in the figure with broken lines G′F′E′K′. Then by counter revolution the base is found as projected.

As the solid is shown, its edges will be perpendicular to the plane of its base, therefore their projections are perpendicular to the traces of the plane of the base. To locate a point in the lower base, as the point D, pass a plane through the edge DK, perpendicular to H and the trace $RR_h$, therefore perpendicular to R. This auxiliary plane cuts the line ML, from R, revolved into H at $M'L_h$. The point K of the base revolves to K″. But ML is perpendicular to the edge KD, being in a plane perpendicular to KD. Then from K″ lay off K″D′ equal to the height of the solid, and counter revolve D′ back to $D_h$. As the upper and lower bases are parallel, the projections of the upper base are drawn parallel to the lower base.

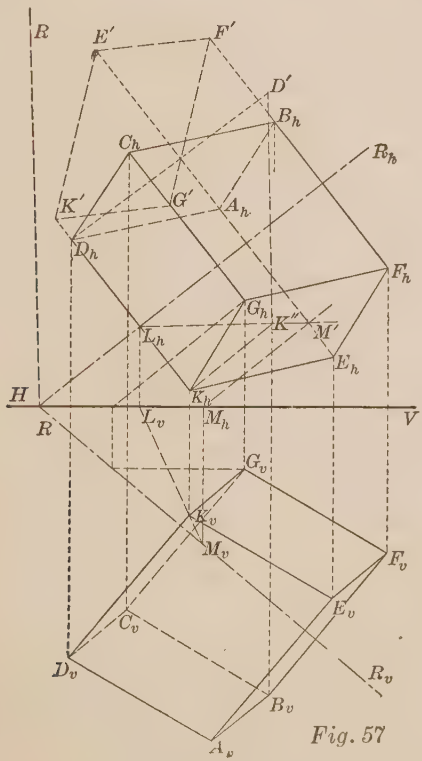



*Fig. 57*

EXAMPLES.

Ex. 1. Given the points A(2.5,—2.0, 0.5), B(4.0,—1.0, —1.5), C(7.0,—1.5,—0.25), D(8.0,—0.25,—1.0), E(3.0,—1.75,

—2.0), F(4.0,—0.25,—1.0), G(6.5, 1.5, 0.25) and K(7.5,—2.5, 1.0). Find the true length of each of the lines AB, CD, EF, GK. Find the point where each line pierces the planes of projection, and the angles which each makes with the planes of projection.

Ex. 2. Given the points A(2.0,—2.0,—3.0), B(4.0,—1.0,—3.0) and C(6.0,—2.5,—3.5). Find the traces of the plane passing through these points. Show the true length of each line and the angles which they make with each other. Bisect the angle CAB, and show the projections of the bisector. Find the angles which AB makes with H, V and P.

Ex. 3. The traces of three planes are given as follows. The plane R passes through the points (4.0, 4, 0, 0.0), (1.0, 0.0, 0.0) and (7.0, 0, 0, 2.75). The traces of the plane S pass through the points (4.0, 4.0, 0.0), (6.0, 0.0, 0.0) and (9.0, 0.0, 4.0), the traces of the plane T pass through the points (3.0, 2.5, 0.0), (9.5, 0.0, 0.0) and (10.0, 0.0, 1.5). It is required to find the projections of the tetrahedron included between these planes and the plane H.

Ex. 4. It is required to find the projections of a right prism, the vertices of the lower base being at the points A(6.0,—1.25,—3.0), B(7.5,—2.25,—4.25), C(8.0,—1.0,—4.0) and D(x,—0.25, —3.0). The plane of the upper base passes through E(10.0,—1.0,—1.5). Find also the projections of the line F(6.0,—1.0,—1.5), G(10.0,—2.5,—2.5) upon the plane of the upper base and find where the line pierces the faces of the prism.

NOTE: While any three points assumed at will in space are sufficient to determine a plane, it does not follow that a fourth point assumed in the same manner will lie in this plane. In order, therefore, to pass a plane through four points, any three may be assumed and their plane found. By the methods of Problem 4, the projections of the fourth point are then found in the plane.

Ex. 5. Pass a plane through A(8.0,—1.0,—1.5) parallel to the lines B(5.25,—1.25,—1.75), C(2.0, 0.0,—2.5) and D(0.0, 2.75, —4.0), E(3.5, 1.75, 0.0). Through F(8.0,—2.5,—3.5) pass a plane perpendicular to DE. Find the intersection GK, of the two planes, and draw a perpendicular from A to GK. Construct the projections of a cube, the perpendicular from A upon GK being one edge, and GK reckoned away from the ground line another.

Ex. 6. The vertical trace of a plane is (6.5, 0.0, 00), (4.0, 0.0, 1.75) and the horizontal trace passes through the point (4.0, —2.75, 0.0). A quadrangular pyramid stands on the plane, the

co-ordinates of the vertices of its base being A (2,0,—2,$\pm$x), B(2.5,—0.75,$\pm$x), C(4.5,—0.5,$\pm$x), and D (3.0,—2.5,$\pm$x). The vertex of the pyramid is at E(6.5,—3.0,—4.5). Show the projections of the pyramid.

Ex. 7. One edge of an octagonal prism passes through the points (0.0,—2.5,—3.75), (11.5,—4.25,—0.5), and one of the edges passes through each of the points A(8.0,—1.25,—1.75), B(8.0,—1.5,—2.25), C(8.0,—1.75,—1.0), D(8.0,—2.25,—2.75), E(8.0,—2.75,—0.75), F(8.0,—3.25,—3.0), G(8.0,—4.0,—2.5). It is required to find the projections of the portion of the prism included between the two planes whose traces pass through the points (0.5, 0.0, 0, 0), (4.5,—4.0, 0.0), (4.0, 0.0,—4.0) and (8.75, 0.0, 0.0), (11.5, 0.0,—1.75), (8.25,—4.5, 0.0).

Ex. 8. A plane R passes through the points A(2, 0, 0), B(6, 0, 3), C(5, 2, 1). A regular pyramid with a square base rests on the R plane with one edge parallel to the H trace and at a distance of one inch from it. The base is 2 inches square, and the height of the pyramid is 3 inches. Draw its projections.

Ex. 9. With the plane of the base assumed as in Ex. 8 draw the projections of a pyramid with a regular hexagon for its base. The center of the base is one inch above H and two inches in front of V. The diameter of the circumscribed circle of the hexagon is 3 inches, the height of the pyramid 4 inches. Draw its projections.

# CHAPTER III.

## Single Curved Surfaces.

### *Classification and Construction of Lines.*

72. Let any point in space be moved in any direction an infinitesimal distance. Then the path of the point is an elementary line. If the point be continued in the same direction the line is a straight line or right line.

If the third position of the point be on any side of the continuation of the path of the point from the first to the second position, the line is a curved line.

If the point be again moved in a new direction, but such that a plane may be passed through the four consecutive positions of the moving point, the path is a curve of single curvature. Then any line lying in a plane, other than a straight or right line, is a curve of single curvature.

If a plane through the first three, or any three consecutive positions of the moving point, fail to contain the next consecutive point, the path is a curve of double curvature.

73. If the plane of a curve of single curvature be parallel to either plane of projection, its projection on that plane shows the true form of the curve, and its projections on the other co-ordinate planes are right lines.

If the plane of a curve of single curvature be oblique to two planes of projection but perpendicular to the third, its projections on the first and second planes will be curved lines, but not of the true form of the curve; while its projection on the third plane is a right line.

If the plane of the curve be oblique to all planes of projection, its projections will all be distorted curves.

74. Since not more than three consecutive points of a curve of double curvature are in the same plane, no plane of projection can be made parallel to more than three consecutive points of the curve. The projection of such a curve on any plane must therefore be a curve of single curvature, and does not show the true form of the curve and can not be a right line.

75. As in right lines, a curve is completely determined when its projections on the three co-ordinate planes are known. Likewise the traces of a curve are found in the same manner as the traces of a right line, as illustrated in Fig. 58.

*Fig. 58*

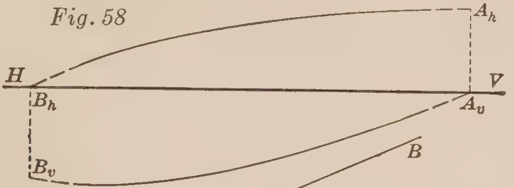



*Fig. 59*

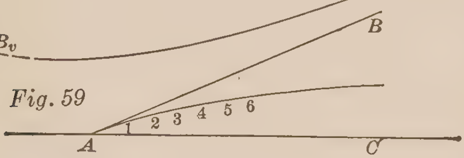
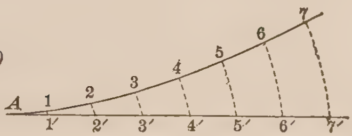



*Fig. 60*




76. Consider a curve made up of infinitesimal right lines, as the lines A1, 12, 23, etc., of Fig. 59, the points A, 1, 2, 3, etc., being consecutive points. The line A1B containing the element of the curve A1, coincides with it through two points and is therefore a tangent to the curve. As A and 1 are consecutive points and therefore at an infinitesimal distance apart, it is convenient to treat the two points as a single point. We would, therefore, speak of AB as tangent at the point A.

Two curves are tangent to each other when they have two consecutive points in common, or when they have a common tangent.

If a right line be tangent to a curve of single curvature, it will lie in the same plane as the curve; for the line has at least two of its points in the plane of the curve, being a tangent.

If a line be tangent to a curve, their projections on the same plane will be tangent.

77. Any right line perpendicular to a tangent at the point of tangency is a normal to the curve. Then an infinite number of normals may be drawn, forming a normal plane. If the curve be of single curvature, the normal is understood to lie in the plane of the curve.

78. If a curve be rolled upon its tangent, the length of any definite portion is shown as a right line, and the curve is said to be rectified, as shown in Fig. 60.

79. The tangent to any curve passes through two consecutive points, and therefore has the same direction in space as that portion of the curve represented by the elementary line joining the consecutive points. The angle which a curve makes with any line is the same as that made by the line with the tangent to the curve. Thus, in Fig. 59, the angle which the curve at A makes with AC is the angle BAC, made by the tangent at A with the line.

## Generation and Classification of Surfaces.

80. A surface may be generated by the continued motion of a line along another line. The moving line is the generatrix, the fixed line being the directrix. Different positions of the generatrix are elements of the surface when the generatrix is a right line. When the generatrix has moved from one point of the directrix to the next consecutive point, the elements are consecutive.

If a right line be moved along another right line, with all positions of the generatrix parallel to its first position, the surface generated is a plane.

If a right line be moved along any line, not a right line, so that its consecutive positions are in the same plane, the surface is a single curved surface.

If a right line be moved so that no two consecutive positions lie in the same plane, the surface generated is a warped surface.

It should be noted that planes, single curved surfaces and warped surfaces are generated by the continued motion of a right line, and therefore have right line elements.

Surfaces which have no right line elements are generated by the continued motion of a curve in some path other than a right line, and are double curved surfaces.

## Single Curved Surfaces.

81. Single curved surfaces are of three kinds:

*First*—Those in which all positions of the generatrix are parallel.

*Second*—Those in which all positions of the generatrix meet in a point.

*Third*—Those in which the consecutive elements intersect but at consecutive points of the directrix, so that not more than two consecutive elements meet in a point.

82. A cylinder may be generated by moving a right line always parallel with itself, but always touching a curve. Or, the curve may be moved parallel with itself, touching a right line.

It is convenient to consider the intersection of the cylinder with the horizontal plane as its base, though the cylinder may have its base in any plane of projection or in any oblique plane.

If the curve of the base has a center, a right line through this point, parallel to the elements, is the axis of the cylinder.

Cylinders may be classified by their bases, as a cylinder with a circular base, a cylinder with an elliptical base, etc.

If the elements be perpendicular to the plane of the base, the cylinder is a right cylinder.

If right lines perpendicular to H, be drawn from consecutive points of any curved line, they form a cylinder which is known as the horizontal projecting cylinder of the line. Similarly, lines perpendicular to V form the vertical projecting cylinder, and perpendicular to P, the perpendicular projecting cylinder. These are convenient auxiliary surfaces in the solution of problems involving curved and warped surfaces.

83. Problem 22. To draw the projections of a cylinder with a circular base and assume a point on its surface.

In Fig. 61 it is required to draw the projections of the cylinder whose base is DGCE and to assume a point, A, on the surface.

Solution. To assume the projections of a cylinder, draw the curve of intersection with either plane of projection and draw the projections of the contour elements.

To assume a point on the surface, assume at will one of its projections and through this point draw the projections of an element.

Construction. In Fig. 61 the base of the cylinder is the circle $C_hD_hE_hG_h$, vertically projected in the ground line at $D_vC_v$. The elements of contour, or the principal elements, are horizontally projected tangent to the base in a direction assumed for the axis. These pass through the points $G_h$ and $E_h$. Their vertical pro-

jections are not contour lines. For the vertical projection of the cylinder, the contour elements pass through the points $C_v$, $D_v$ horizontally projected at $D_h$,$C_h$.

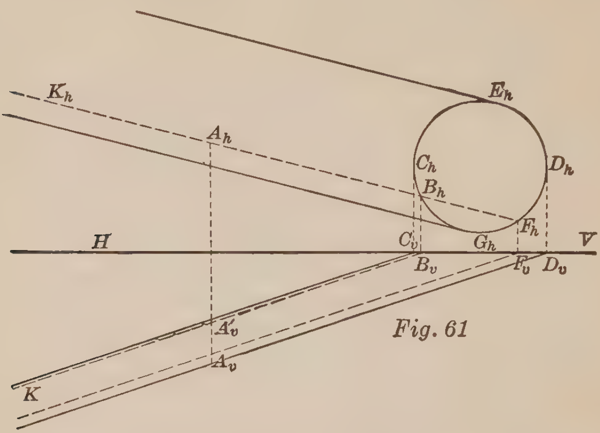



*Fig. 61*

To assume a point A, on the surface, assume at will its horizontal projection, as $A_h$, and draw through this point the horizontal projection of an element intersecting the base at $B_h$ and $F_h$. Then two points of the cylinder may be horizontally projected at $A_h$, for two elements are projected in $K_h A_h B_h F_h$, one being the line $K_hB_h$, intersecting H at $B_h$, on the upper side of the cylinder. The other is on the under side of the cylinder, intersecting H at $F_h$. The vertical projection of FA is $F_v A_v$, and if on the under side, A will be vertically projected at $A_v$. If on the upper side, its vertical projection is $A'_v$.

Any element of a cylinder may be assumed by drawing the projections of a line parallel to the contour elements, intersecting the projections of the base.

84. Problem 23. To pass a plane tangent to a cylinder through a given point on the surface.

In Fig. 62 it is required to pass a plane through the point A of the surface, tangent to the cylinder.

Solution. If the plane be tangent to the cylinder at a point, it will be tangent all along the element through the point. The element will be a line of the tangent plane and will pierce the planes of projection in the traces of the tangent plane. If the cylinder and tangent plane be intersected by any plane, the line of intersection of the two planes will be tangent to the curve cut

from the cylinder; and if the cutting plane be one of the planes of projection, the line will be the trace of the tangent plane in that plane.

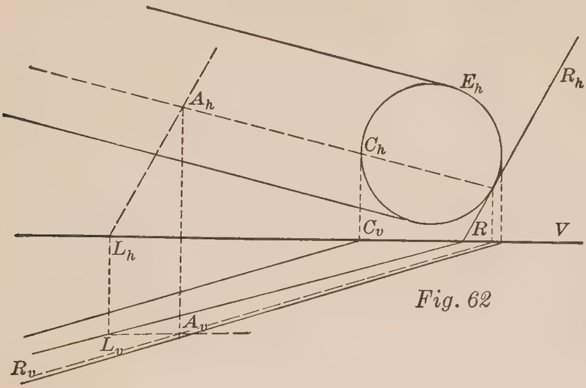



*Fig. 62*

CONSTRUCTION. If an element be drawn through the given point it will be a line of the required tangent plane. A tangent to the base, at the point where this element pierces H is the horizontal trace. The vertical trace may be drawn through the vertical trace of the element of tangency. Or, through any known point of the plane, not in H, a line may be drawn parallel to the horizontal trace. The horizontal projection will be parallel to the horizontal trace and its vertical projection will be parallel to the ground line.

85. PROBLEM 24. To pass a plane through a given point, without a cylinder, tangent to the cylinder.

In Fig. 63 it is required to pass a plane through A, tangent to the cylinder whose base is EDG.

SOLUTION. Since a plane which is tangent to a cylinder is tangent all along an element, the problem is simply to pass a plane through this tangent element and the given point.

CONSTRUCTION. If a line be drawn through the given point, parallel to the elements of the cylinder, it will pierce H in a point of the required horizontal trace. A line through this point, tangent to the base, is the horizontal trace. The vertical trace will be found by producing the line through the given point to pierce the vertical plane, locating a point in the vertical trace, or by the method shown in Fig. 62, drawing an auxiliary line of the required plane through a point in the known line, parallel to the horizontal trace.

Two planes may always be passed through a point without a cylinder, tangent to the cylinder. In Fig. 63, the planes R and S pass through the point A and are tangent to the cylinder.

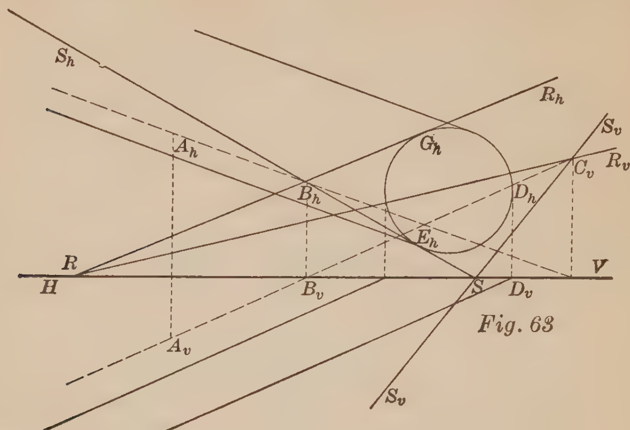



*Fig. 63*

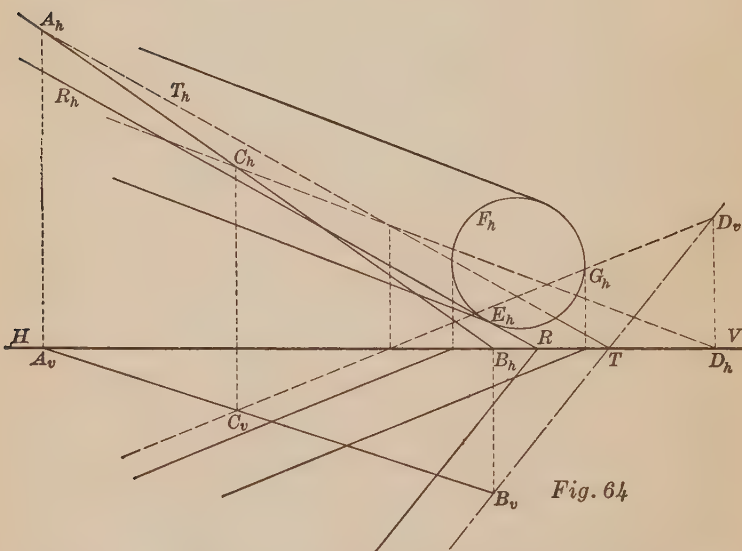



*Fig. 64*

86. Problem 25. To pass a plane tangent to a cylinder, parallel to a given right line.

In Fig. 64 it is required to pass a plane parallel to the line AB, tangent to the cylinder whose base is EFG.

Solution. If through any point of the given line a line be drawn, parallel to the elements of the cylinder, they will form a

plane which will be parallel to the required plane. The horizontal trace of the required plane may, therefore, be drawn tangent to the base of the cylinder, parallel to the horizontal trace of this auxiliary plane. The planes being parallel, their vertical traces will be parallel.

CONSTRUCTION. Through any point C, of the line AB, draw CD parallel to the elements of the cylinder. These lines have $TT_h$ as the horizontal trace of their plane. The horizontal trace of the tangent plane is $RR_h$ tangent at $E_h$ and parallel to $TT_h$. A second plane may be passed tangent to the cylinder.

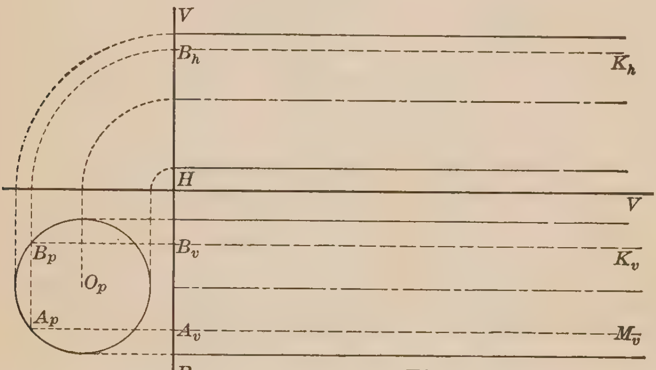


*Fig. 65*

87. In these problems, if the axis of the cylinder be parallel to the ground line HV, the solution is only possible by means of the third plane of projection. For example, let it be required to draw the projections of an element of a cylinder whose axis is parallel to the ground line HV, as in Fig. 65.

The perpendicular projection of the cylinder is a circle, shown at $A_pB_p$. Assume at will the horizontal projection of an element, as $B_hK_h$. But two elements of the cylinder may be horizontally projected in $B_hK_h$, as shown by their perpendicular projections in $B_p$,$A_p$. The vertical projections of these elements are shown at $A_vM_v$,$B_vK_v$.

88. Let it be required to pass a plane tangent to a cylinder through a given point on the surface, when the axis of the cylinder is parallel to the ground line HV.

Returning to Fig. 65, let A be the given point. The element AM will be the element of tangency. This element pierces the P plane at $A_p$, a point in the perpendicular trace. Tangent to the

circle (the base of the cylinder in P) at $A_p$ draw the perpendicular trace. Complete the construction by drawing the traces of the tangent plane in H and V.

Ex. 1. Through a given point without the cylinder pass a plane tangent to the cylinder, when its axis is parallel to the ground line HV.

Ex. 2. Pass a plane tangent to a cylinder, parallel to a given right line, when the axis of the cylinder is parallel to the ground line.

## Cones.

89. A cone is a single curved surface generated by a right line, always touching a curve, and always passing through a fixed point not in the plane of the curve.

If the point be in the plane of the curve, one limit of the cone has been reached, and the surface generated is a plane.

If the point be at an infinite distance from the curve and not in its plane, another limit of the cone has been reached and the surface is a cylinder.

The generatrix being a right line of indefinite length, intersecting for all its positions in a fixed point, the vertex of the cone; whatever surface is generated on one side of the vertex will have a duplicate on the opposite side. These portions of the complete surface thus generated are called *nappes*.

Any intersecting plane not passing through the vertex will cut a curve from the cone, which may be used as a directrix for generating the surface.

The intersection of the horizontal plane with the elements of the cone is considered its base, though its base may be in any plane of projection or in any oblique plane.

As in the case of cylinders, cones are designated by their bases; as a cone with a circular base, a cone with an elliptical base, etc.

90. Let any plane of symmetry be passed through the vertex, intersecting the cone. Then if the elements diametrically opposite with respect to such a plane, make the same or equal angles with the plane, the cone is a right cone.

If the elements all make the same angle with the axis the cone is a right cone with a circular base, and may be generated by the revolution of the hypothenuse of a right triangle about either side as an axis.

91. PROBLEM 26. To draw the projections of a cone with a circular base and assume a point on its surface.

In Fig. 66 it is required to assume a point A on the cone whose base is the circle CBD and whose vertex is the point K. All elements of a cone pass through the vertex. Then to assume the

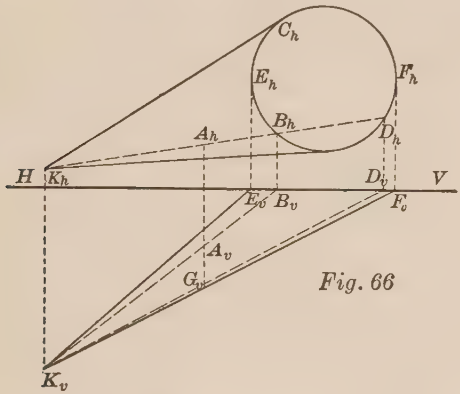



*Fig. 66*

horizontal projection of the cone, tangents are drawn from K to the horizontal projection of the base, these, as in cylinders, being the contour elements when the eye of the observer is above H. The contour elements for the vertical projection pierce H at $E_h$, $F_h$ and the vertical projection is shown at $K_vF_vE_v$.

To assume a point, as the point A, on the surface, assume at will one projection, as the horizontal at $A_h$. Draw the horizontal projection of an element $K_hA_h$. This cuts the projection of the base in $B_h$ and $D_h$, showing that two elements of the surface may contain the point A, one on the upper side piercing H at $B_h$ and the other on the lower side, piercing H at $D_h$. The vertical projections of these points are at $A_v$, $G_v$.

92. PROBLEM 27. To pass a plane tangent to a cone through a given point on the surface.

In Fig. 67 it is required to pass a plane tangent to the cone through the point A.

SOLUTION. If through the given point an element of the cone be drawn, it will be a line of the tangent plane, and will pierce H in a point of the horizontal trace. This trace will be tangent to the base at the point where the element of tangency pierces the plane of the base.

The vertex of the cone being in every element of the cone

will be a point of every plane, tangent to the cone. Then to find the vertical trace it is convenient to draw a line through the vertex, parallel to the horizontal trace of the tangent plane and its vertical trace will be a point in the V trace of the plane. Or, the element of tangency may be produced to pierce the V plane, giving a point in the required V trace.

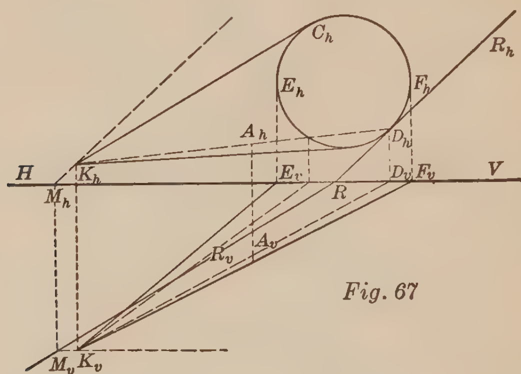



Fig. 67

93. Problem 28. To pass a plane tangent to a cone through a given point without the surface.

In Fig. 68 it is required to pass a plane through the point A, tangent to the cone.

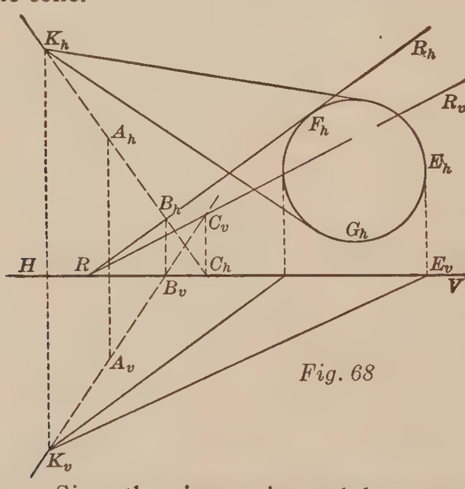



Fig. 68

Solution. Since the given point and the vertex are known to lie in the tangent plane, a line joining them will be a line of the tangent plane and will pierce H and V in points of the traces. The horizontal trace will be drawn tangent to the base and through the horizontal trace of the line.

CONSTRUCTION. The line KA pierces H at $B_h$ and V at $C_v$. Through $B_h$ draw the horizontal trace tangent to the base of the cone. The vertical trace is drawn through $C_v$.

From $B_h$ a second tangent may be drawn to the base of the cone, showing that a second tangent plane may be passed through the given point.

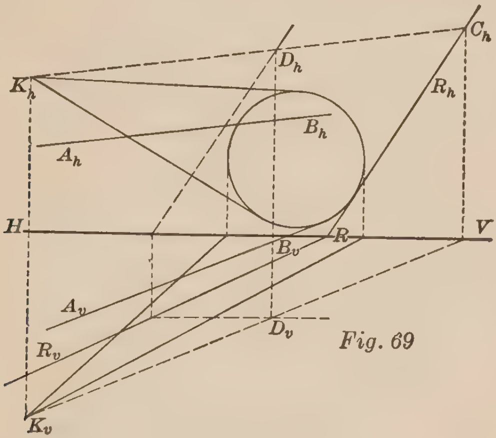



*Fig. 69*

94. PROBLEM 29. To pass a plane tangent to a cone, parallel to a given right line.

In Fig. 69 it is required to pass a plane tangent to the cone, parallel to the line AB.

SOLUTION. Since the vertex is a point in the required tangent plane, a right line through the vertex, parallel to the given line, will be a line of the required plane. From the point in which it pierces H the horizontal trace may be drawn tangent to the base.

CONSTRUCTION. A line through K, parallel to AB, pierces H at $C_h$. Through $C_h$ draw the horizontal trace tangent to the base.

Two lines may be drawn through $C_h$ tangent to the base; therefore two planes may satisfy the conditions of the problem.

If the line through the vertex, parallel to the given line, pierce the horizontal plane within the limits of the base, no tangent to the base can be drawn and the problem is impossible.

## EXAMPLES.

Ex. 1. Draw the projections of a cone and assume an element of the surface, when the axis of the cone is parallel to the ground line HV.

Ex. 2. Pass a plane tangent to a right cone through a given

point without the surface, when the axis of the cone is parallel to the ground line HV.

Ex. 3. The base of an oblique cone is in the P plane. Pass a plane tangent to the cone through a given point without the surface.

Ex. 4. Pass a plane tangent to a cone, parallel to a given right line, when the axis of the cone is parallel to the ground line HV.

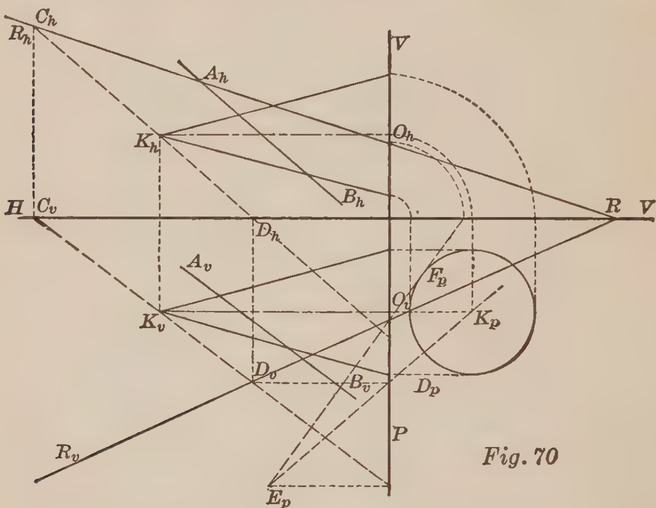



*Fig. 70*

The construction of Ex. 4 is given in Fig. 70, in which AB is the given line and the axis KO of the cone is parallel to the ground line HV. The line CD drawn through K pierces V at $D_v$, a point in the vertical trace of the required plane. It pierces H at $C_h$. But as the base is in the P plane, it is necessary to draw the P projections, both of the cone and auxiliary line, through its vertex. $K_p$ is one point in the P projection of CD. The point D being in V, is perpendicularly projected in the ground line VP. Then $K_pD_p$ is the projection of CD on P, piercing P at $E_p$, a point in the P trace of the required tangent plane. Draw $E_pF_p$ tangent to the base of the cone, this being the P trace of the plane R. The point of intersection of $E_pF_p$ with the VP ground line is a point common to the V and P traces, and is therefore a second point in the required V trace of the tangent plane R.

## Convolute Surfaces.

95. Single curved surfaces of the third class are known as

convolute surfaces and may be generated by moving a right line so that it shall always be tangent to a curve of double curvature.

In generating such a surface any two, but not three consecutive elements will lie in the same plane; for they are the extensions of the elementary lines of the directrix, of which any two but not three consecutive elements intersect and therefore lie in a plane. An infinite number of such surfaces may be generated, for an infinite number of lines of double curvature may be drawn. For the solution of problems it is convenient to use the helix, this being a familiar curve and one easily drawn.

96. PROBLEM 30. To draw the projections of a convolute surface and assume a point on the surface.

Let the directrix be the helix, as shown in Fig. 71 at ABCDEF. To draw the base of the surface and to assume a point, as the point M.

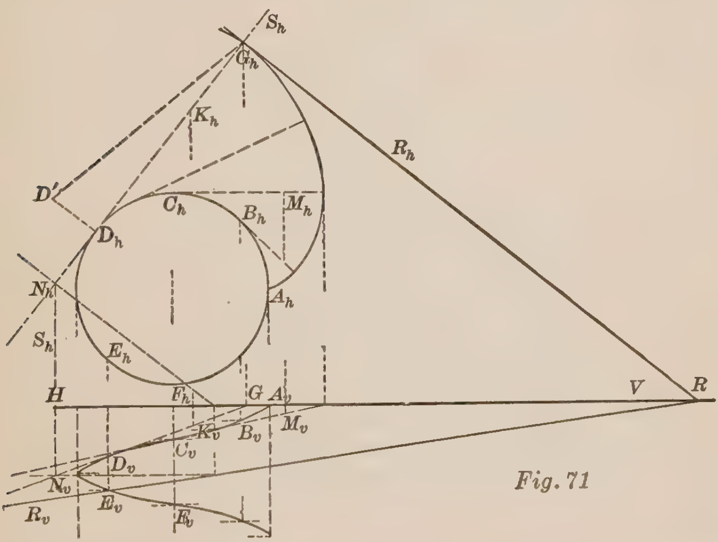


*Fig. 71*

SOLUTION. Since the surface is made up of tangents to the helix, to find the base of such a surface it is only necessary to find the intersections of the tangents with H.

Let the H plane pass through A, a point of the helix. Then A is in the base of the surface and is the beginning of the curve of the base. Consider the helix drawn on a cylinder by moving a point at a uniform rate along the axis, while it moves at a uniform rate and at a constant distance (the radius of the cylinder)

about the axis. Let the projections of the helix be shown in ABCDEF, Fig. 71, starting at A in the H plane. Pass any horizontal projecting plane tangent to the cylinder and therefore tangent to the helix at D, as the plane SS. Develop the cylinder on this plane. To show this development, revolve the tangent plane about its H trace. The horizontal projection of the helix from A to D will be rectified in $D_hG_h$, being made equal in length to the arc $A_hB_hC_hD_h$. The point D of the helix is at a distance below H equal to the distance of $D_v$ from the ground line—shown in the revolved positions in $D_hD'$. The line $D'G_h$ is the rectified curve of the helix from A to D. The angle $D'G_hD_h$ is the angle made by the helix with H and is therefore equal to the angle made by all tangents to the helix with H. $D'G_h$ may be considered a tangent to the helix at D, piercing H at the point G at a distance from $D_h$ equal to the arc $A_hB_hC_hD_h$. Then to find any point in the curve of the base, draw any tangent to the horizontal projection of the helix. Lay off along the tangent a distance equal to the rectified arc from the assumed point of tangency to the origin of the curve. The curve of the base is recognized as the involute of a circle. The vertical projections of the elements of the surface may be found in the usual manner, by noting that the point of tangency to the helix and the point in which an element pierces H are known. The vertical projections are omitted from Fig. 71 in order to simplify the drawing. To assume any point in the surface, as the point M, its horizontal projection is assumed at will and the projections of an element drawn.

It should be noted that while the surface extends indefinitely in all directions, it has no points inside the horizontal projection of the helix. As the elements are indefinite in extent, they may be produced both ways from the points of tangency, making the surface above any assumed point on the helix similar to that below.

In Fig. 71 the element through M is taken with its horizontal projection parallel to the ground line. Then its vertical projection shows the true angle which the elements make with H.

97. Problem 31. To pass a plane tangent to a convolute surface, through a given point on the surface.

Let the surface be the single curved surface with the helical directrix, as given in Fig. 71, and let K be the given point.

Solution. The required plane will be tangent to the surface

along the element through the given point, and its horizontal trace will be tangent to the base at the point where this element pierces H.

CONSTRUCTION. The element through K pierces H at G. The normal to the curve of the base at G is the horizontal projection of the element GD, for G is at the instant revolving about D, the point of tangency on the helix. The vertical trace may be found by drawing through a known point of the plane, as N, a line parallel to the H plane, its H projection being parallel to the H trace and its V projection parallel to the ground line HV. Or, a point in the V trace may be found by producing the element GD to pierce the V plane.

98. PROBLEM 32. To pass a plane tangent to the convolute surface through a given point without the surface.

In Fig. 72 it is required to pass a plane through the point M, tangent to the convolute surface whose directrix is the helix ABGK.

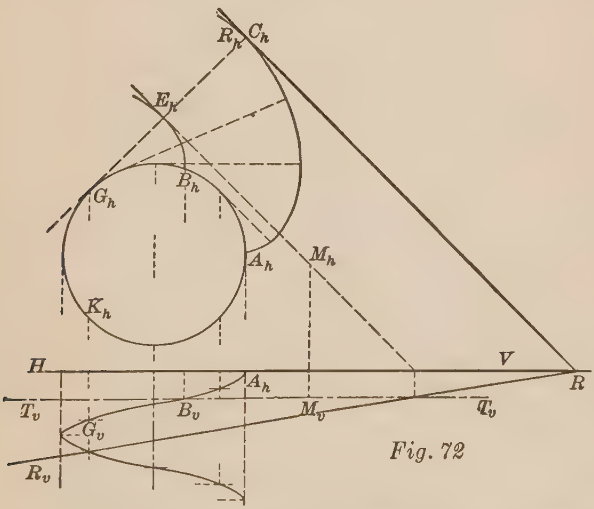



*Fig. 72*

SOLUTION. If a new horizontal plane be passed through the given point and the base of the surface be found upon this auxiliary plane, a line from the given point, tangent to this new base, will be a line of the required tangent plane. This line is the trace of the required plane on the auxiliary horizontal plane. The element of tangency pierces the auxiliary plane at the point of tangency. The element thus drawn and the line through the given point determine the tangent plane.

CONSTRUCTION. The horizontal plane T is passed through M. This plane is pierced by the helix in the point B. Then $B_h$ is the origin of the curve of intersection of the convolute surface with the plane T, the curve being found in the same manner as the curve of the base, $A_hC_h$, etc. Draw $M_hE_h$ tangent to the curve $B_hE_h$. Then ME is a line of the required plane. From $E_h$ draw $E_hG_h$ tangent to the helix. This will be the element of tangency, and therefore a second line of the plane R.

As the plane T is parallel to H, the line ME in T and also in the required plane R will be parallel to the required horizontal trace of R.

99. PROBLEM 33. To pass a plane tangent to a convolute surface and parallel to a given right line.

In Fig. 73 it is required to pass a plane tangent to the convolute surface whose base is $A_hL_hM_h$ and parallel to the line GK.

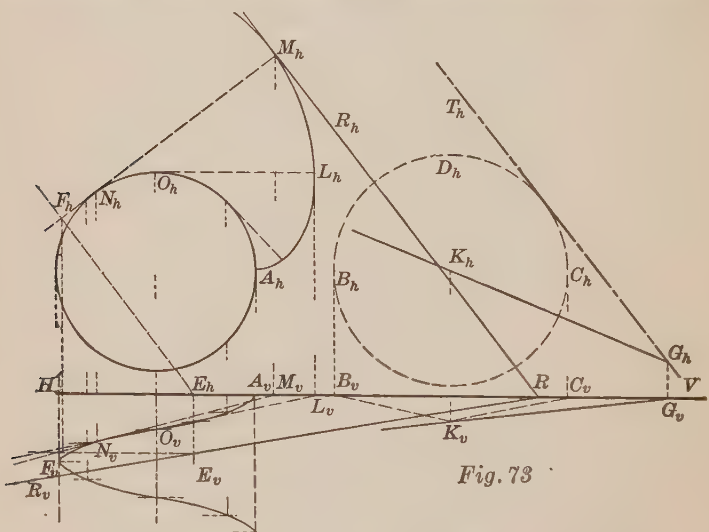


Fig. 73

SOLUTION. If a right cone be constructed with its vertex at any point of the given line, whose elements make the same angle with H that is made by the elements of the convolute surface, and a plane be passed through the given line, tangent to the cone, it will be parallel to the required tangent plane.

CONSTRUCTION. Draw the element OL, of the convolute surface, parallel to V, therefore in its vertical projection showing the angle which all the elements make with H. With any point of GK, the given line, as a vertex, construct an auxiliary cone

with its elements making the angle $O_vL_vE_h$ with H. The plane T tangent to this cone and through the given line, will be parallel to the required plane. A point in the vertical trace of the plane T may be had by producing the line GK, to pierce V. Or, as in the figure, a point in the vertical trace of R may be found by drawing the element of tangency and through any point as F, draw an auxiliary line of the plane.

If the given line pierce the horizontal plane within the base of the auxiliary cone, no tangent can be drawn to the base, and the case is impossible.

When the given line pierces the horizontal plane outside the base of the auxiliary cone, two tangents may be drawn, and therefore two planes may be passed tangent to the given surface.

## CHAPTER IV.

### Double Curved Surfaces.

100. Double curved surfaces have no right line elements and are, therefore, generated by the motion of curved lines in paths other than right lines.

If the curve be rotated about any right line as an axis, the surface will be a surface of revolution, and the section cut from such a surface by a plane perpendicular to its axis, will be a circle.

While it is not necessary that a double curved surface should be a surface of revolution, only such surfaces will be considered here, as the engineer seldom has to deal with any other.

Two surfaces of revolution, and only two, can be single curved surfaces; viz., a right line revolving about another right line, with which it is parallel, generating a cylinder; and a right line revolving about another right line with which it intersects, generating a cone.

A right line revolving about another right line with which it is not parallel and which it does not intersect, generates a warped surface known as the hyperboloid of revolution of one nappe. This is the only surface of revolution which is a warped surface. It will be studied in the chapter devoted to warped surfaces.

101. The double curved surfaces of revolution commonly met with are the following:

If a circle be revolved about any diameter as an axis, the surface generated will be the sphere.

If an ellipse be revolved about its major axis as an axis, the surface generated will be the prolate spheroid.

If an ellipse be revolved about its minor axis as an axis, the surface generated will be the oblate spheroid.

If the parabola be revolved about its axis the surface generated will be the paraboloid.

If the hyperbola be revolved about its transverse axis, the surface generated will be the hyperboloid of revolution of two nappes.

If the hyperbola be revolved about its conjugate axis, the surface generated is the hyperboloid of revolution of one nappe.

As stated above, this surface may also be generated by revolving a right line about another right line, with which it is not parallel and does not intersect.

If a curve be revolved about any right line, not an axis of symmetry of the curve, the surface generated will be the torus.

The usual form of the torus is generated by revolving a circle about an axis, not a diameter, but lying in the plane of the curve, the center of the circle being at a distance from the axis greater than its radius.

102. Any surface of revolution may be regenerated by the motion of a circle along an axis through its center and perpendicular to its plane, by regulating the diameter according to some fixed law. In the case of the cylinder the diameter remains constant; in all other cases the diameter changes.

103. If a double curved surface of revolution be intersected by a plane passing through its axis, the curve cut from the surface will be a meridian curve and the plane will be a meridian plane.

The meridian curve which is cut by a plane parallel to V is the *principal meridian.*

If a meridian plane be passed through two surfaces which intersect and have a common axis, their meridian curves will intersect in two points. These points, being in the same perpendicular to the axis, when revolved will generate a circle which is common to both surfaces, and is therefore their intersection. If the meridian curves have more than one point of intersection on the same side of the axis, there will be more than one circle of intersection.

If the meridian curves be tangent at two points, one on each side of the axis, their axes coinciding, these points in revolution will generate a circle which is common to both surfaces and is therefore their line of contact.

104. It is usual to represent a double curved surface by taking its axis perpendicular to H. The vertical projection will be the meridian curve parallel to V, therefore the principal meridian. The horizontal projection will be as many concentric circles as are needed to show the contour.

105. If a plane be tangent to a double curved surface, at

least one point of the surface will lie in the plane. For, if any plane, other than the tangent plane, be passed through the point, it will cut from the tangent plane a right line and from the surface a curve, which will be tangent at the point.

As any number of planes may be passed through the point of contact of a plane with a double curved surface, each cutting a right line from the plane and a curve from the surface, which are tangent to each other; it follows that the tangent plane is the locus of all tangents drawn to the surface at the point of contact.

106. Since any two lines determine a plane (if they intersect or are parallel), to pass a plane tangent to a double curved surface at a given point, draw two lines of the surface intersecting at the point; the tangents to these lines, at their common point, determine the tangent plane.

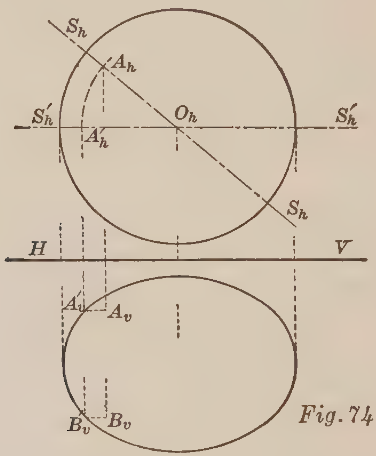


Fig. 74

107. A right line is normal to a surface at a point, when it is perpendicular to the tangent plane which passes through the point. Any plane which contains this normal will be perpendicular to the tangent plane, and will be known as a normal plane. When the solid is a sphere the meridian plane passing through the point of contact of a tangent plane, contains the radius to that point. But the radius is normal to the surface at the point of tangency, and as it lies in the meridian plane through the point of tangency, the meridian plane is perpendicular to the tangent plane.

108. PROBLEM 34. To draw the projections of a double curved surface, and to show the projections of a point on the surface.

In Fig. 74 let the surface be the oblate spheroid. To find the projections of a point on the surface, as the point A.

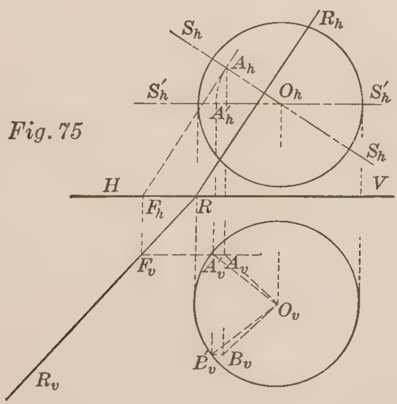
SOLUTION. If the meridian plane passing through the point be revolved about the axis of the surface to a position parallel to V, the vertical projection of the point will be found in the principal meridian of the surface. After counter revolution the true projection will be shown.

CONSTRUCTION. Pass the meridian plane S through $A_h$, assumed at will anywhere on the horizontal projection of the surface. The plane S will be perpendicular to H and will contain the axis of the surface. Revolve S about the axis O, until it is parallel to V. The meridian curve passing through A will then be vertically projected in the principal meridian, and the point may be at $A'_v$ or $B'_v$. Revolving the plane S back to its original position, the vertical projection will be at $A_v$ or $B_v$.

109. PROBLEM 35. To pass a plane tangent to a sphere at a given point on the surface.

In Fig. 75 it is required to pass a plane tangent to a sphere at the given point A.

Fig. 75

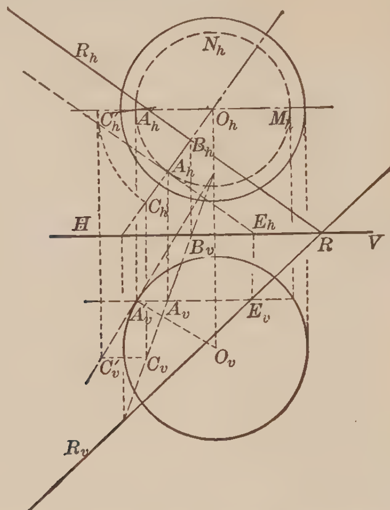



SOLUTION. Since the radius of the sphere, drawn to the point of contact of the tangent plane, is perpendicular to that plane, it is only necessary to pass a plane through the given point, perpendicular to the radius drawn to that point.

CONSTRUCTION. Through the point $A_h$, draw $A_hF_h$, perpendicular to the horizontal projection of the radius AO. The line

AF, being parallel to H, its vertical projection will be parallel to the ground line and its horizontal projection will be parallel to the required horizontal trace.

Second Method. SOLUTION. If tangents be drawn to any two lines of the surface, intersecting in the given point, the plane of these tangents will be the required plane.




CONSTRUCTION. In Fig. 76, when the meridian curve of the surface through the point A, is revolved parallel to V, the tangent at A is shown in the line $A'_vC'_v$. The tangent intersects the axis of revolution at a point which remains fixed. After counter revolution the tangent is projected at $A_v$ $C_v$, $A_h$ $C_h$, piercing H at $B_h$. A plane through A, parallel to H, cuts a circle from the sphere, horizontally projected in $A_hN_hM_h$. A tangent to this circle at the point A is a second line of the tangent plane, piercing V at $E_v$, a point in the required vertical trace. The line $E_hA_h$ will be parallel to the required horizontal trace of the tangent plane, being parallel to H.

110. PROBLEM 36. To pass a plane tangent to any surface of revolution at a given point on the surface.

Let the given surface be the ellipsoid of revolution, as shown in Fig. 77, and let A be the assumed point on the surface.

SOLUTION. The tangents drawn to any two curves of the surface through the given point will be lines of the required tangent plane.

CONSTRUCTION. Pass the meridian plane S through the given

point A. To draw a tangent to the meridian curve cut from the surface through the point A, revolve the plane S about the axis OO until it is parallel to V. The meridian curve will then be vertically projected in the principal meridian. A tangent to this curve at $A'_v$ will be the revolved position of a line of the required plane. This tangent pierces H at $C_h$, a point in the horizontal trace. A second curve of the surface is the circle $A_h$ $M_h$. A tangent to this is a second line of the tangent plane.

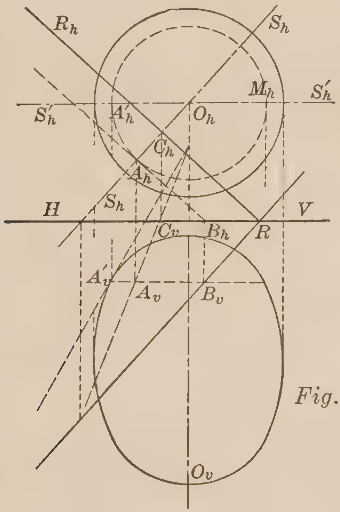


Fig. 77

111. Problem 37. To pass a plane tangent to a sphere through a given right line.

In Fig. 78 it is required to pass a plane tangent to the sphere through the given line AB.

Solution. If a plane be passed through the center of the sphere, perpendicular to the given right line, it will be pierced by the given line in a point, will cut from the sphere a great circle; and the line drawn from the point in which the given line pierces the plane, tangent to the circle cut from the sphere, will be a line of the required plane.

Construction. The plane R is passed through O, the center of the sphere, perpendicular to the line AB, and is pierced by AB in the point A. (Construction for finding A not shown.) . The plane R, passing through the center of the sphere, cuts from it a great circle. To draw this circle and a tangent to it from A, the plane R is revolved into H about its horizontal trace. (This construction is not given in the figure.) The center O, falls at

O′ and the point A at A′. With center O′ and radius equal to the radius of the sphere, draw the great circle, cut from the sphere by the plane R. The tangent A′E′ to this circle is a line of the required tangent plane. After counter revolution the tangent is at AE. Through AE and AB the plane may be passed.

Much of the construction of this problem is omitted because the figure would be too complex if all were included.

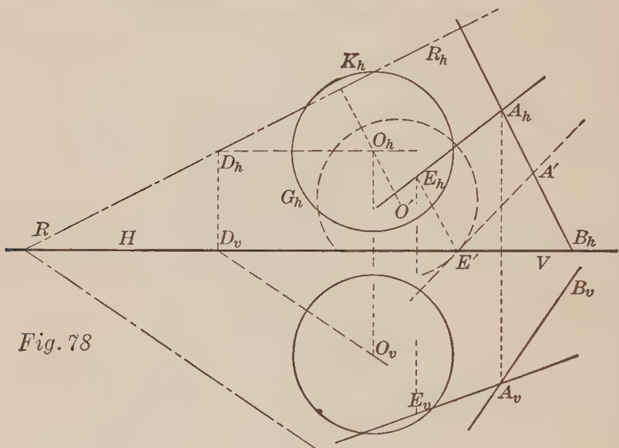


Fig. 78

112. Second Method. Problem 37.

Solution. If, with any point of the given line as a vertex, a cone be constructed with its elements tangent to the sphere, a plane tangent to this cone through the given line will be tangent to the sphere.

Construction. Let A of the line AB be the vertex of a cone, tangent to the sphere. The circle of contact of the sphere with the cone may be considered its base, and the plane of this circle the plane of its base. If the point be found in which the given line AB, pierces this plane and a tangent be drawn from this point to the base, the tangent will be a second line of the required tangent plane.

In Fig. 79, AO is the axis of the cone. The horizontal projecting plane of AO cuts a great circle from the sphere and two elements from the cone, also a diameter from the circle of tangency of the cone and sphere. The horizontal projecting plane is revolved into H, A falling at A′, O at O′; the great circle cut from the sphere is shown incomplete in F′G′. The tangents to the great circle are A′F′ and A′G′. The diameter of tne circle of tangency is the line G′F′, M′ being the center of that circle. After counter revolution, M′ is horizontally projected at $M_h$, vertically

at $M_v$. The plane of the base, R, passes through M perpendicular to the axis AO. The line AB pierces R at B. If, now, the plane R be revolved into H about its H trace, B will fall at B′ and M at M″. With center M″, and radius M′F′, draw the circle N′K′. From B′ draw B′K′ tangent to this circle, this being the trace of the required tangent plane in the plane R. After counter revolution, this tangent is projected in the line BK. Through AB and BK pass the required tangent plane.

Much of the construction is omitted from Fig. 79 to avoid a complex drawing.

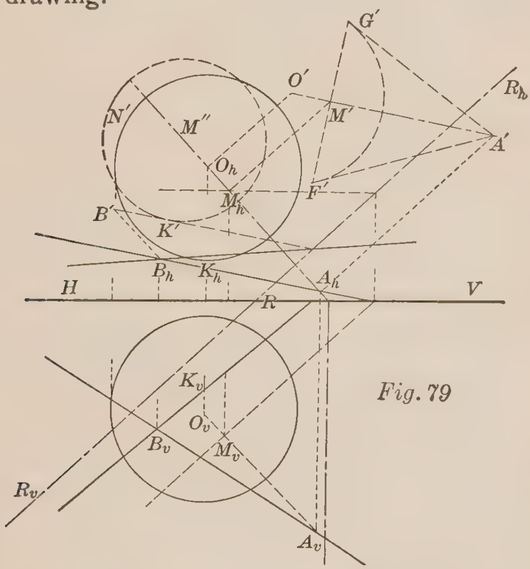



*Fig. 79*

113. A discussion may be given here of a problem which is usually given in the chapter on the point, line and plane, but requires the use of a sphere as an auxiliary surface in its solution.

PROBLEM 38. Let it be required to draw the traces of a plane when the angles made by the plane with two planes of projection are given.

In Fig. 80 let B be the angle made by the plane with H and A the angle made by the plane with V.

SOLUTION. Construct a sphere of any radius with its center in the ground line HV, and assumed to be tangent to the required plane. If, through the center of the sphere a plane be passed perpendicular to the horizontal trace of the required plane, it will be perpendicular to that plane and to H; and will cut from the required plane a line which is tangent to the circle cut from the

auxiliary sphere and makes with H the angle which the plane makes with H. A similar construction with reference to the vertical trace determines a second line of the required plane.

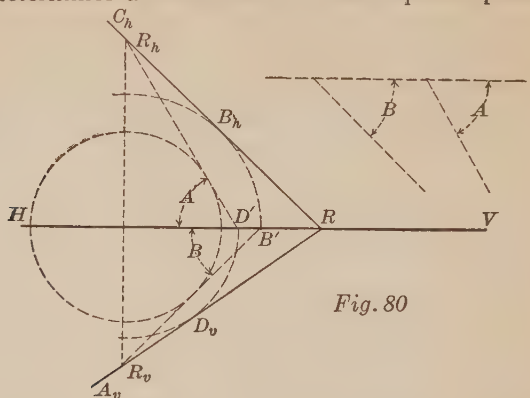



*Fig. 80*

CONSTRUCTION. With the center of the auxiliary sphere in the ground line, its vertical and horizontal projections coincide. If a plane be passed through the center of the sphere, perpendicular to the required horizontal trace, and then be revolved about its vertical trace into the vertical plane, the circle cut from the sphere will be vertically projected in the projection of the sphere. The line cut from the required plane, shown at $A_vB'$, will be tangent to this circle, and the angle which it makes with the ground line is equal to the angle which the plane R makes with H. The line $A_vB'$ pierces V in $A_v$, a point in the V trace of the required plane, and pierces H somewhere in the circular path $B'B_h$. Similarly, by passing a plane through the center of the sphere and perpendicular to the vertical trace of the required plane, and revolving this plane into H, the angle made with V is the angle which the line $D'C_h$ makes with the ground line. DC pierces H at $C_h$ and V somewhere in the arc $D'D_v$. Draw through $C_h$ the horizontal trace of R, tangent at $B_h$ to the arc $B'B_h$. In a similar manner the vertical trace is drawn.

If a plane be parallel to the ground line HV, the sum of the angles made by the plane with H and V will be 90°. If the plane be perpendicular to the ground line HV, the sum of the angles made with H and V will be 180°, each being 90°. Then the sum of the angles which a plane makes with H and V must be as great as 90° and not greater than 180°.

Ex. 1. Through the point A(4,—3,—2) pass a plane which makes with H an angle of 75° and with V an angle of 45°.

## CHAPTER V.

### INTERSECTION OF SINGLE AND DOUBLE CURVED SURFACES BY PLANES, AND THE DEVELOPMENT OF SURFACES.

114. If it be required to find the point in which a line pierces a surface, it is only necessary to pass a plane through the line, intersecting the surface. This auxiliary plane will cut a line from the surface, which will intersect the given line in the required point.

The auxiliary plane should be selected which cuts the simplest possible line or lines from the surface. Thus, if the surface be a cylinder, the plane should be parallel to its axis. If the surface be a cone, the auxiliary plane should pass through its vertex. If the surface be a sphere, the auxiliary plane should be passed, parallel to one of the planes of projection, preferably through the center of the sphere.

If the given line be a curve of single curvature, its plane will intersect the surface in a line, which, if there be an intersection, will intersect the given line in the required point, or points.

If the given line be a curve of double curvature, a cylinder or cone may be constructed as an auxiliary surface, which will contain the curve on its surface. The intersection of this auxiliary surface with the given surface will, if there be an intersection, contain the required point, or points.

115. To find the line of intersection of a plane with a surface, intersect the plane and surface by a system of auxiliary planes. Each auxiliary plane will cut from the given plane a line, and from the surface either a right line or a curve, which will intersect in a point or points of the required line of intersection. The line of intersection is drawn by joining the points of intersection.

116. The tangent at any point of the curve of intersection of a plane with a surface will lie in the cutting plane, and will also be in the plane tangent to the surface at the assumed point. Therefore, to draw a tangent to the curve of intersection of a

plane with a single curved surface, pass a plane tangent to the surface at the assumed point. The intersection of the tangent plane with the cutting plane will be the required tangent.

117. If a single curved surface be rolled upon the tangent plane passed through any element of the surface, until its right line elements consecutively lie in the tangent plane, the surface is developed, and the plane has the same area as the surface of the solid.

Only surfaces having two consecutive elements in the same plane can be developed, as only such surfaces can be made to coincide with a tangent plane. Therefore, only single curved surfaces can be developed.

In order to find the positions of the elements when a surface is developed, it is necessary to choose some curve of the surface, which, when rolled upon the tangent plane, will develop in a right line, an arc of a circle, or some simple curve which can be correctly drawn. Upon the development of this curve the rectified distances between elements may be laid off.

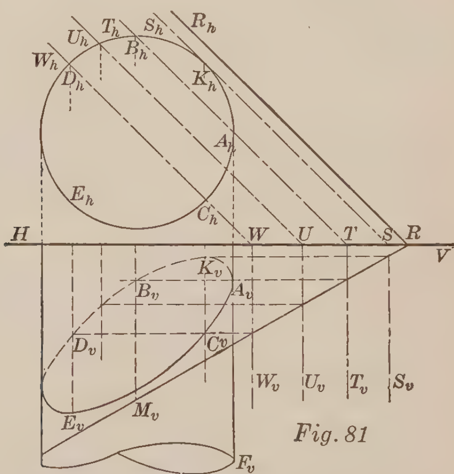


*Fig. 81*

118. Problem 39. To find the intersection of a right cylinder with a circular base by an oblique plane.

In Fig. 81 it is required to find the curve of intersection of the cylinder whose base is ADCB, with the plane R.

Solution. If a system of auxiliary planes be passed parallel to the axis of the cylinder, right lines will be cut from the plane

and elements from the cylinder, which intersect in points of the required curve of intersection.

CONSTRUCTION. If these auxiliary planes be passed parallel to the horizontal trace of R, they will cut lines from the plane which are horizontally projected in the horizontal traces of the auxiliary planes, and vertically projected parallel to the ground line. For example, the plane T cuts the line AB from the plane R, and the elements FA and MB from the cylinder. These elements intersect the line AB in the points A and B, two points in the curve of intersection. Other points are determined in like manner and the curve of intersection drawn.

In selecting auxiliary planes care should be taken to locate the critical points in the curve. For example, the plane S locates the highest point K of the curve. Another plane should locate the lowest point; others should pass through the contour elements of the cylinder to determine the visible portion of the curve, etc.

Second Method. SOLUTION. If planes be passed perpendicular to the axis of the cylinder, they will cut circles from the cylinder and right lines from the plane. The intersection of these lines and circles are points of the required curve. The vertical projections of these circles are in the vertical traces of the auxiliary planes.

119. PROBLEM 40. To draw a tangent at any point to the curve cut from a cylinder by an oblique plane.

In Fig. 82 it is required to draw a tangent to the curve of intersection of R with the cylinder, through the point G.

SOLUTION. If a plane be passed through the assumed point tangent to the cylinder, its line of intersection with the cutting plane will be the required tangent line.

120. PROBLEM 41. To develop that portion of a right cylinder with a circular base lying between H and an oblique plane.

In Fig. 82 it is required to find the development of the portion of the cylinder lying between the planes R and H.

SOLUTION. Since the plane of the base is perpendicular to every element of the cylinder, it is perpendicular to the two consecutive elements which lie in a tangent plane; and as all the elements are parallel, the base will roll out in a straight line. On this line the rectified lengths of arc of the base, between any two elements, may be shown, and the elements be drawn their true lengths parallel to the element of tangency.

Construction. The curve of intersection is found by Problem 39. Let the element CK be the element of tangency, shown in the developed position at C′K′. From C′ lay off the rectified distances C′1′, 1′2′, etc., equal to the arcs $C_h1_h, 1_h2_h$, etc. Since the axis of the cylinder is perpendicular to H, the base is a right section and may be conveniently rectified in the ground line. The heights C′K′, L′1′, etc., are the portions of the elements between the planes. The area between the curve K′G′N′K′, and the rectified base, and between the elements K′C′ and K′C′ is the required development.

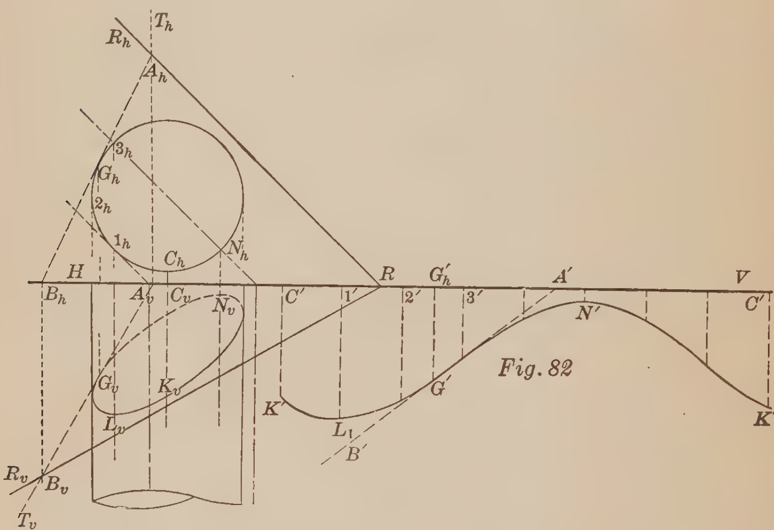


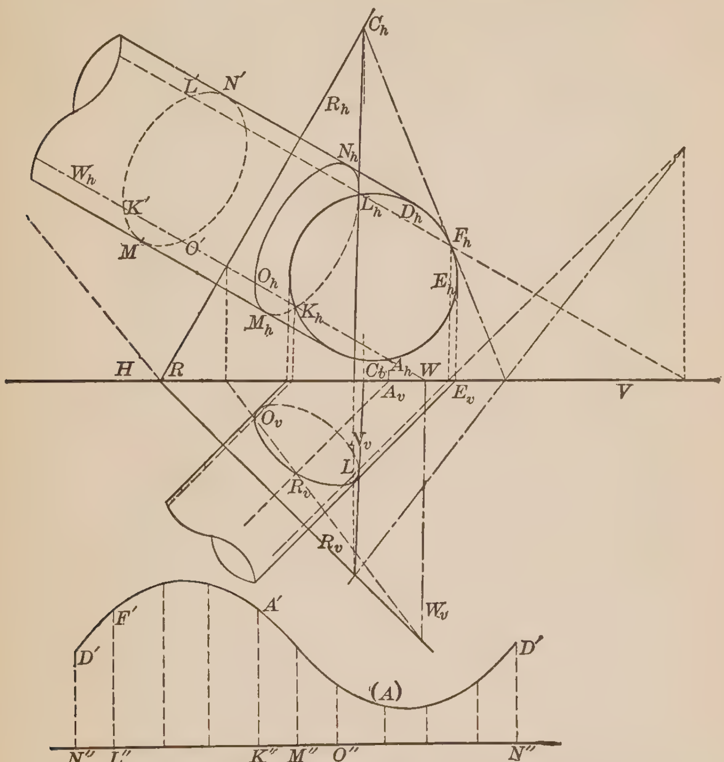

*Fig. 82*

In getting the projection of the tangent AB on the developed surface, it should be noted that the tangent is the hypothenuse of a triangle of which the base is the projection $G_hA_h$, and the altitude the distance from $G_v$ to HV. Lay off from $G'_h$ the distance $G_hA_h$ to A′ and draw the tangent G′A′.

121. Problem 42. To find the intersection of an oblique cylinder by any plane, to draw a tangent to the curve of intersection and to develop that portion of the cylinder between H and the cutting plane.

In Fig. 83 it is required to find the curve of intersection of the cylinder with the plane R and to develop the portion of the cylinder between H and R. Also to draw a tangent to the curve of intersection at the point L.

Solution. Intersect the cylinder and plane by any system of planes which will cut elements from the cylinder. Each plane (except the tangent planes which cut one each) will cut two elements from the cylinder and a line from the plane, which intersect in points of the required curve of intersection.




To draw a tangent at any point of this curve, pass a plane through the point tangent to the cylinder. The intersection of the tangent plane with the cutting plane will be the required tangent line.

To develop the surface, pass a plane perpendicular to the axis, cutting a right section from the cylinder. As this section when rolled out on a tangent plane will develop into a right line, distances between elements may be shown on it. Distances from this section to any other, as the base, may be measured along the elements by first finding their true lengths and laying off the distance between the right section and the base.

Construction. The simplest planes to use in this case are

either perpendicular to H or V. The plane W, passed parallel to the axis of the cylinder and perpendicular to H, cuts two elements from the cylinder and a line from the plane, locating the points K and O of the curve. Other points are located in a similar manner, and the curve MKLNO drawn. As the auxiliary planes are parallel, their intersections with the plane R are parallel lines. The portions of the curve behind the contour elements in the vertical projection and below the contour elements in the horizontal projection are dotted, because hidden by the cylinder. Care should be taken to note the visible and invisible points of the curve as the auxiliary planes are used.

The plane R in Fig. 83 is assumed perpendicular to the axis of the cylinder. This is done that the section will be a right section and may be used in the development of the surface. Otherwise another plane, perpendicular to the axis, would be required, which would make the drawing very complex.

To find the true form of the right section cut by the plane R revolve R into H, the true form showing in the curve M′K′L′N′. If the cylinder be developed on a tangent plane, this section will develop into a straight line, shown at (A) Fig. 83, in the line M″L″K″N″. On this line lay off the rectified distances between elements, M″K″, K″L″, etc. On the perpendicular through K″ lay off the true distance from K to A, a point in the required curve, found by revolving the line KA parallel to V or H. The points D, F, etc., are found in like manner, locating the curve D′F′A′D′, the development of the intersection of the cylinder with H.

In general, to draw a tangent at any point of the curve of intersection of an oblique cylinder with a plane after development, note that the true length of the element of tangency, shown in the development, is one side of a triangle, the line in H joining the point where the tangent pierces H with the point where the element of tangency pierces H is a second side, and the third is the true length of the tangent from the point of tangency to where it pierces H. This triangle is constructed on the developed position of the element of tangency.

In Fig. 83, the plane being perpendicular to the axis of the cylinder, in the development its curve of intersection coincides with the tangent.

122. It should be noted that in Fig. 83 the intersection of

the horizontal plane with the cylinder, which is oblique, is a circle. It therefore follows that a right section can not be a circle. Another section, cut by a plane making the same angle with the axis as is made by the plane H, but in an opposite direction, will also be a circle. Such a section is known as a sub-contrary section.

123. PROBLEM 43. To find the intersection of a right cone with a circular base by an oblique plane, to show the curve of intersection in its true form, and to draw a tangent to this curve at any point.

In Fig. 84 it is required to find the intersection of the plane R with the cone whose base is ABCG, and to draw a tangent at the point M.

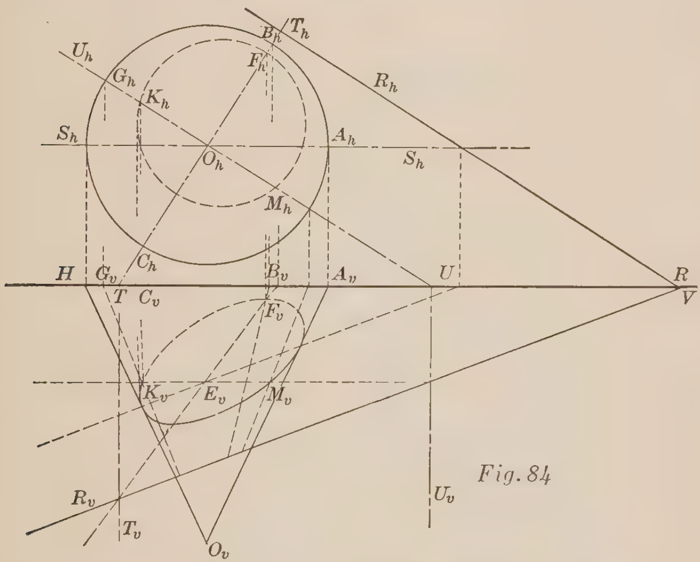


*Fig. 84*

SOLUTION. If planes be passed through the vertex of the cone, perpendicular to the horizontal plane, they will cut from the plane right lines, and elements from the cone, which intersect in points of the required curve of intersection.

To draw a tangent at any point, pass a plane tangent to the cone through the assumed point. It will intersect the given plane in the required tangent.

To find the true form of the curve of intersection it is only necessary to revolve the plane of the curve into H or V.

CONSTRUCTION. Since all the auxiliary planes, T, U, etc.,

are perpendicular to H and pass through a common point O, the vertex of the cone, they intersect in a common line, the axis of the cone. This line pierces the plane R in the point E. Then all the lines cut from R by the auxiliary planes pass through the point E. To draw additional auxiliary planes, as the plane S, it is only necessary to draw the horizontal trace. The vertical projection of the intersection of the horizontal traces of the planes R and S, is in the ground line, and with $E_v$ will determine the required line.

The drawing of the tangent at the point M and finding the true form of the curve of intersection of R with the cone are omitted from the figure.

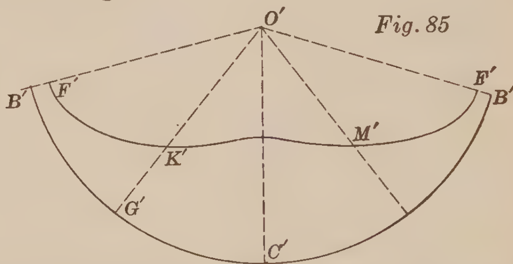


*Fig. 85*

124. In Fig. 84 if the plane R intersect the cone and be passed parallel to an element, the curve cut from the cone is the parabola.

If the cutting plane be parallel to the axis of the cone, the curve of intersection is the hyperbola, and in general will be a curve of two nappes.

If the plane cut the cone at an angle between parallel to an element and perpendicular to the axis, the curve of intersection is the ellipse.

If the plane be perpendicular to the axis, the curve cut from the cone is a circle.

125. PROBLEM 44. To develop the right cone with a circular base.

Let the cone be given as in Fig. 84. To develop the portion of the cone above the plane R.

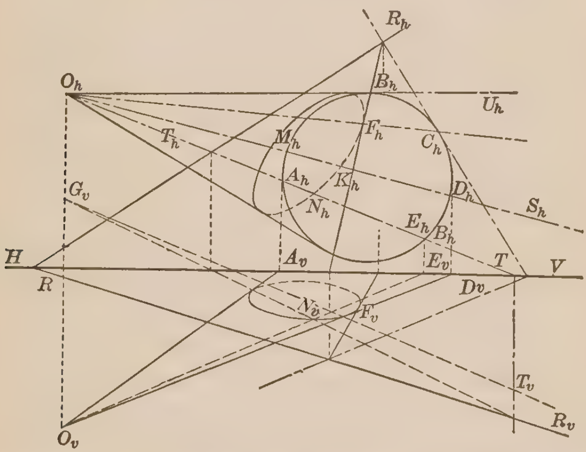
SOLUTION. Since the cone is a right cone with a circular base, every point in the base is equally distant from the vertex. Therefore if an arc of a circle be drawn with a radius equal to the slant height of the cone, the center will be the developed position of the vertex, and a portion of the circumference the

developed base. Measuring on this circumference a length equal to the base and drawing the elements of the cone completes the development. Lay off from the vertex the true distances to the curve of intersection, locating points in the developed curve.

CONSTRUCTION. To find the true length O'F', it is necessary to first revolve OF until it is parallel to V. Other points in the curve are found in a similar manner and the curve drawn, as shown in Fig. 85.

126. PROBLEM 45. To find the intersection of any oblique cone by a plane.

In Fig. 86 it is required to find the intersection of the plane R with the cone whose base is ABCD.




SOLUTION. If through the vertex of the cone planes be passed perpendicular to H, they will cut elements from the cone and right lines from the plane. These intersect in points of the required curve of intersection.

CONSTRUCTION. The planes S, T, U, etc., being perpendicular to H, and passing through the vertex of the cone, intersect in a line perpendicular to H, through the point O. This line will pierce the plane R, in a point common to all the auxiliary planes, and is therefore a point in the line of intersection of R with each of these planes. This common point is G, horizontally projected at $O_h$, the vertex of the cone. The points of the curve are determined by methods of Problem 43.

127. PROBLEM 46. To develop an oblique cone, to draw a

tangent to any point of the curve of intersection of a cone by an oblique plane, and to find the true section cut by such a plane.

Let the cone be given as in Fig. 86. To draw a tangent to the curve KFM, at any point, as F, to show the true form of the curve, and to develop the portion between H and R. The development is shown in Fig. 87.

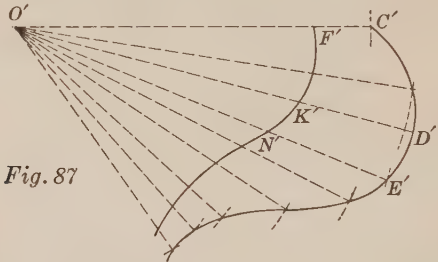



*Fig. 87*

Solution. Since no section made by a plane with an oblique cone will make the same angle with all the elements, points in the base will not be equally distant from the vertex, and hence can not develop in the arc of a circle. To develop an oblique cone, construct a triangle of which one side is the rectified base between any two elements, the other sides being the true lengths of these elements. Construct a second triangle on one of these developed elements as a side, the rectified base between this and a new element being a second side, and the true length of the new element a third. Continue this construction till enough points are found to draw the developed curve of the base. From the vertex lay off along the elements the true distances to the cutting plane, locating points in the curve of intersection.

To show the true form of the curve of intersection of the cutting plane with the cone, revolve the plane into one of the planes of projection.

The intersection of a tangent plane through any point on the curve of intersection of a cutting plane, with the cutting plane, will be the tangent to the curve at that point.

Construction. In Fig. 87, with O′, the vertex of the cone, as a center, and true length of the element EO, as a radius, strike an arc through E′. Draw the element O′E′. Similarly, with center O′, and radius O′D′, the true length of the element OD, strike an arc through D′. With center E′, and radius equal to the rectified arc of the base from E to D, strike an arc locating the point D′. In a similar manner construct the triangle O′D′C′, etc., until all the base is developed. Points in the curve F′K′N′

are found by laying off from O′ the true lengths OF, OK, etc. In the figure only a part of the development is shown.

The student will show the true form of the section made by the plane R, with the cone in Fig. 86, and draw a tangent to the curve at the point F.

128. PROBLEM 47. To find the curve of intersection of a plane with a convolute surface, as the single curved surface with the helical directrix.

In Fig. 88 it is required to find the curve of intersection of the plane R with the surface whose base is the curve ABCD.

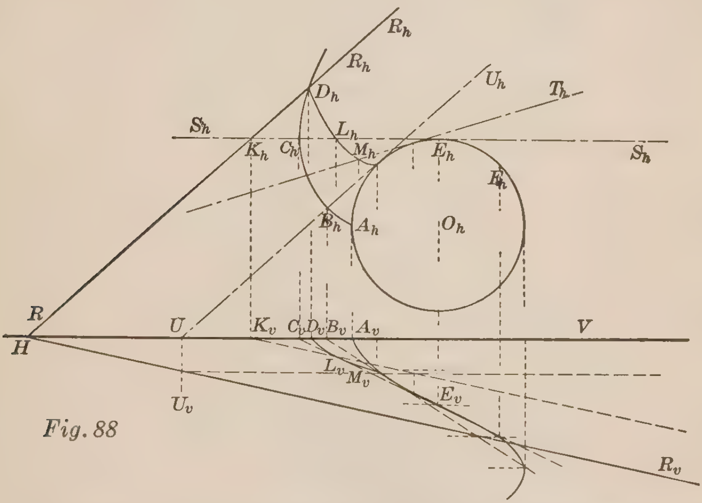


Fig. 88

SOLUTION. If auxiliary planes be passed tangent to the helical directrix, perpendicular to the horizontal plane, they will cut elements from the surface, and right lines from the plane, which intersect in points of the required curve of intersection.

CONSTRUCTION. The auxiliary plane S cuts the line KL from the plane R, and the element CE from the surface, their horizontal projections coinciding in the trace of the plane S. They intersect in L, a point in the curve of intersection. Similarly, the plane T locates the point M. The horizontal projection of the curve does not cross the horizontal projection of the helix, as no points can be horizontally projected inside the circle with the elements tangent to it.

129. PROBLEM 48. To draw the curve of intersection of a double curved surface by a plane.

In Fig. 89 it is required to find the curve of intersection of the plane R with the ellipsoid of revolution.

SOLUTION. First Method If auxiliary planes be passed through the surface of revolution, parallel to H, they will cut circles from the surface of revolution and right lines from the plane, which intersect in points of the required curve of intersection.

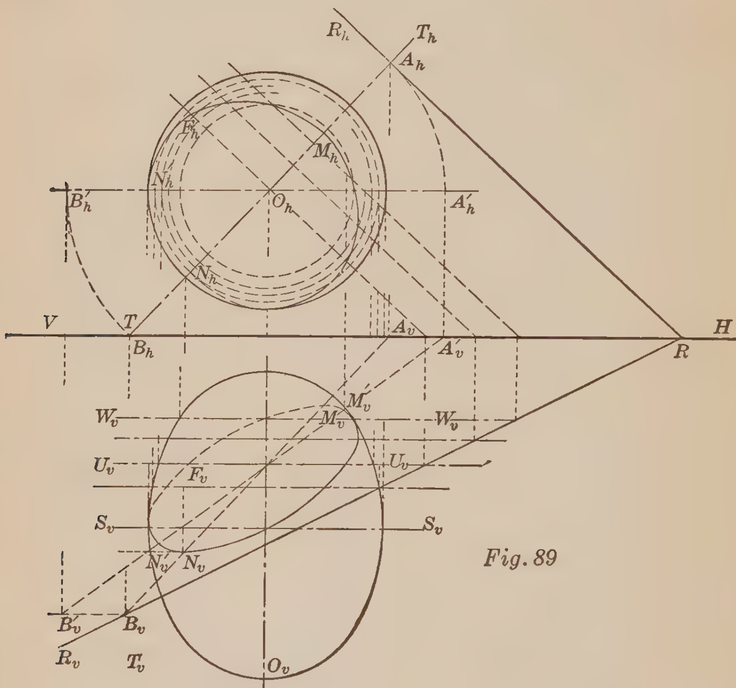


Fig. 89

CONSTRUCTION. First Method. Since the auxiliary planes are parallel to H, the horizontal projections of the circles cut from the surface will show in their true form. Moreover, the lines cut from the plane R by the auxiliary planes will be horizontally projected parallel to the horizontal trace of R. The true size and shape of the section cut from the surface may be shown by revolving R parallel to H or V.

It is convenient to know at the start between what limits the auxiliary planes parallel to H should be passed, thus locating the highest and lowest points in the curve. This may be done by passing the meridian plane T, perpendicular to the horizontal trace of R. The line of intersection of R and T is a line of sym-

metry of the curve cut from the solid, piercing it at the highest point M, and the lowest point N, of that curve.

Second Method. If meridian planes be passed through the surface they will cut meridian curves from the surface and right lines from the plane, which intersect in points of the required curve of intersection. To locate these points the meridian planes are revolved about the axis of the surface until they are parallel to V and their meridian curves are therefore projected on V in the principal meridian.

## CHAPTER VI.

### INTERSECTION OF SOLIDS.

130. The intersection of any two solids may be found by passing through them a system of auxiliary surfaces, usually planes. Each auxiliary surface, if it intersects both the given surfaces, will determine one or more points of the curve of intersection.

The system of auxiliary surfaces should be chosen to cut the simplest possible lines from the given surfaces.

131. Let it be required to draw a tangent to the curve of intersection at any point, the given surfaces being curved surfaces. Since the point is on each surface, pass a plane tangent to each surface through the given point. The line of intersection of these tangent planes is the required tangent line.

132. In constructing the curve of intersection, great care should be taken to determine which points are visible to the observer, and to so mark them that there will be no mistake in joining projections of points to get the projections of lines. As an aid in drawing these projections, the critical points of the curve should be found first, such as the highest and lowest points, the points where the curve becomes invisible, the points located by the planes which are tangent to one or both surfaces, etc.

133. PROBLEM 49. To find a curve of intersection of any two single curved surfaces.

Let the surfaces be a cylinder and cone, as given in Fig. 90.

SOLUTION. If the surfaces be intersected by a system of planes passing through the vertex of the cone and parallel to the axis of the cylinder, they will cut elements from both surfaces which intersect in points of the required curve of intersection.

CONSTRUCTION. In Fig. 90 the line OE passing through the vertex of the cone and parallel to the axis of the cylinder is a line of every auxiliary plane, and pierces H in a point common to every horizontal trace, and V in a point in every V trace. The plane S has its H trace through the common point for all H

traces, and is drawn tangent to the base of the cylinder. The V trace is drawn through the common point of all V traces. It is tangent to the cylinder along the element KB and cuts two elements from the cone, which pierce V at 2, 3. The element KB of the cylinder intersects the elements cut from the cone at G and K. It should be noted that if a plane be passed tangent to the cone at the point K, its intersection with the plane S, tangent to the cylinder, will be the element of the cone through K. But this makes the element through K also the tangent to the curve

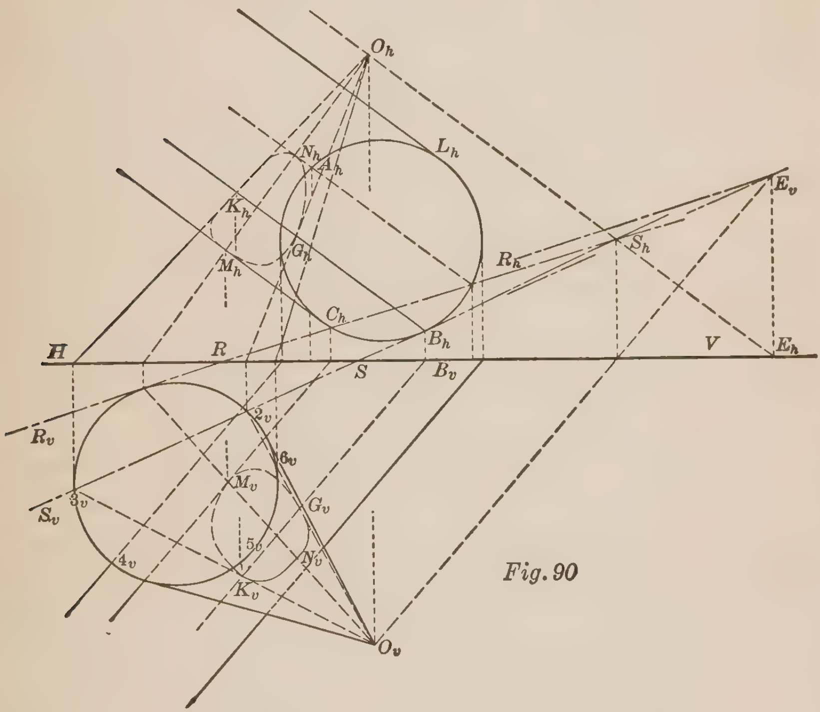


*Fig. 90*

of intersection through K; for the elements of the cone intersect elements of the cylinder between G and K. The element through K does not, therefore, cut the curve, and since both are on the surface of the cone, it must be tangent. Then the direction of the curve through K, also through G, is known, and the curve should be sketched tangent to elements of the cone through these points. In like manner the plane R, tangent to the cone, cuts two elements from the cylinder, locating the points M, N. The curve should again be sketched, tangent to the two elements cut

from the cylinder. With the curve sketched through the points M, N, K, G, its general shape is known. In the figure, it will be a single closed curve. If the planes R and S were both tangent to the cone, this would show that the cone passed through the cylinder and the intersection would then be two closed curves. Similarly, if the limiting planes were both tangent to the cylinder, it would pass through the cone, giving two closed curves.

Additional auxiliary planes are passed through the surfaces, between R and S, locating enough points to draw the curve. These planes should be selected to locate the critical points of the curve, as the highest points, points determined by contour elements of the surfaces, etc.

Care should be taken to determine the visible and invisible portions of the curve in the two projections. In the horizontal projection of the cylinder the elements are visible which pierce H along the curve of the base $C_h A_h$ to $L_h$, and the elements of the cone are visible which pierce V along the curve of the base, $3_v$ $2_v$ to $6_v$. Then in the horizontal projection, if elements intersect which pierce H and V between these limits, the point is visible. Continuing the investigation, the student will determine the visible portion of the curve of intersection in the vertical projection.

As the planes are passed through the surfaces, all the points of intersection should be found and properly lettered or marked, and where possible, the curve sketched as they are found.

134. If the surfaces be cylinders, the system of auxiliary planes will be passed parallel to their axes.

If both the given surfaces be cones, the system of auxiliary surfaces will intersect in the line which joins the vertices of the cones.

## Examples.

Ex. 1. An oblique cone with a circular base stands on H, the center of the base being O(5.0,—2, 0.0); radius of base is 1.5″, and the vertex of the cone at (8.0,—4.0,—4.0). It is required to draw the projections of the cone, the projections of its intersection by the plane which passes through the points A(8.0, 0.0, 0.0), B(2.0, 0.0,—4.0) and C(3.0,—4.0, 0.0), and to find the true form of the curve by revolving it to the right into H. Also develop the portion of the cone between H and the cutting plane.

At any convenient point in the curve of intersection draw a tangent and show the tangent after development.

Ex. 2. A cylinder with a circular base stands on H, with its center at O(4.5,—2.75, 0.0), radius 1.25″, the elements being parallel to the line A(5.5,—0.75,—5.5), B(0.75,—4.0, 0.0). A cone with a circular base also stands on H with its center at the point C(6.0,—2.50, 0.0), radius 2″, vertex at the point D(5.5,—0.75,—3.5). It is required to construct the projections of the surfaces, and of their curve of intersection, and to draw a tangent to the curve at any convenient point.

Ex. 3. A sphere with a radius 1.5″ has its center at A(3.5, —1.5,—1.5). A cylinder with a circular base, radius 1.25″, stands on H, center of base at B(x,—1.5, 0.0); the elements are parallel to V, and inclined to the right at an angle of 60° to H, and the extreme left-hand element is tangent to the sphere. It is required to draw the projections of the surfaces with their curve of intersection.

Ex. 4. A convolute surface is formed by tangents to a helix, the center of its horizontal projection being at the point O(4.5,—2.0, 0.0), the radius of this projection being 1.25″, and the angle made by the curve with H, 45°. The surface is intersected by a plane with a trace passing through the points A(1.0, 0.0, 0.0), B(4.0,—4.0, 0.0) and the plane makes with H an angle of 45°. It is required to draw the projections of the curve of intersection.

Ex. 5. It is required to find the intersection of two cylinders. The first stands on H with its center at the point O(4.0,—3.0, 0.0), radius 1.5″, axis parallel to V but inclined 30° to the right of a vertical line. The second cylinder, 3 inches in diameter, has its axis parallel to the ground line 2 3/4″ behind V, and its top element in H. It is also required to develop both cylinders.

Ex. 6. Given a sphere with radius 2″, center at O(3.0, —2.25,—2.0), and a plane through A(6.5, 0.0, 0.0) perpendicular to V and inclined to H. The section of the sphere by the plane forms the base of a cone, whose vertex is at the lowest point of the vertical diameter of the sphere. It is required to find the true form of a section of this cone by a plane parallel to H, through the center of the sphere.

Ex. 7. A circle 1.5″ in diameter, with its plane perpendicular to H, revolves about an axis which is also perpendicular to H, and at a distance of 3″ from the center of the circle, gener-

ating a torus. It is required to find the intersection of this surface by an oblique plane, and to show the true form of the curve or curves of intersection.

Ex. 8. To find the intersection of two right cones, one base being in H and the other in P. The base in H is 2″ in diameter and the base in P 3″ in diameter, and the elements nearest V intersect.

Ex. 9. To find the position of a guide pulley and its shaft, to take a belt in either direction from one pulley to another, whose shafts are at right angles to each other. The limitation is the well-known principle that a belt must be delivered from a pulley, into the plane of the pulley which is to receive it.

Ex. 10. Given the elevation of the outline of a twelve-sided vase, to draw its complete projections, and to develop one of the sides as a pattern.

Ex. 11. A cylinder 2″ in diameter has its axis perpendicular to P and passes through the point (1.0,—1.5,—2.0). A prism with a regular hexagon as its base in H, has its edges parallel to the line A(1,—3, 0), B(4, 0,—3). The center of the circumscribed circle of the base is at (0.0,—3.0, 0.0), radius 3″. Find the curve of intersection of the cylinder with the prism and develop both surfaces.

Ex. 12. A cast-iron water pipe 24″ in diameter is entered by a pipe 14″ in diameter, their axes making an angle of 75°, and their top elements intersecting. Draw the projections of the curve of intersection.

Ex. 13. A cylinder with a circular base has its axis passing through the points A(2.0, 3.0, 0.0), B(4.0, 2.0, 3.0), the diameter of the base being 1.5″. A second cylinder has its axis passing through the points B and C(7.0, 4.0, 0.0). These cylinders are to be made in sheet metal. Find the projections of the second cylinder of such dimensions and section that it will exactly fit the first. Show the true sections and develop both cylinders.

Ex. 14. Find the right section of the raking moulding which will mitre with a gutter, the pitch of the roof and the section of the gutter being given.

Ex. 15. Draw the projections of a five-piece elbow to be made in sheet metal. Diameter of cylinder three inches. Develop one of the pieces.

# CHAPTER VII.

## Warped Surfaces.

135. A warped surface is generated by a right line moving so that no two consecutive positions are either parallel or intersect; therefore so that no two consecutive elements of the surface lie in the same plane.

Warped surfaces are represented by projecting the directrices and several elements, or, what is the same thing, showing several positions of the generatrix.

An infinite number of surfaces may be generated by the continued motion of a right line, which by definition will be warped surfaces. The more common ones only can be treated here.

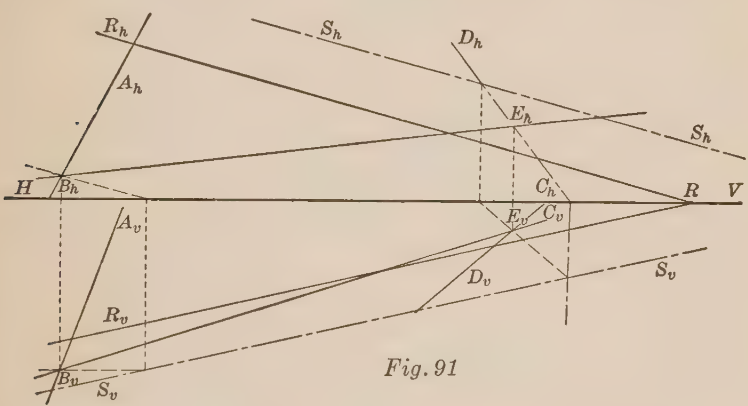



*Fig. 91*

136. A right line moving along two lines, not in the same plane, but always moving parallel to a given plane, will generate a warped surface. The plane is known as the plane director and the fixed lines as the directrices.

If the directrices be right lines, the surface is known as the hyperbolic paraboloid.

If one directrix be a right line and the other a curve, the warped surface will be known as the conoid. If the right line directrix be perpendicular to the plane director, the surface is the right conoid, and the right line is known as the line of stricture.

137. Problem 50. To draw the projections of an element of a warped surface when the directrices and the plane director are given.

In Fig. 91 let R be the plane director and AB and CD the directrices.

Solution. If through any point of one directrix an auxiliary plane be passed parallel to the plane director, it will cut from the required surface an element, which will intersect the second directrix in the point in which that directrix pierces the auxiliary plane.

Construction. The simplest possible solution of this problem is shown in Fig. 91, where the directrices are right lines AB and CD. Through B of the line AB, a plane S is passed parallel to the plane director R. This auxiliary plane S is pierced by the directrix CD, in the point E. But E and B are points in the plane S, parallel to the plane director, and the line joining them will therefore be parallel to the plane director. The points B and E, are also points of the directrices and therefore lie in the surface. The line BE, therefore, is an element of the warped surface.

138. If the general case of the problem be assumed, the construction is not so simple; for instead of the horizontal projecting plane of the second directrix CD, Fig. 91, we must, in general, deal with the horizontal projecting cylinder of CD, and the line of intersection of the auxiliary plane, and this cylinder, must be found.

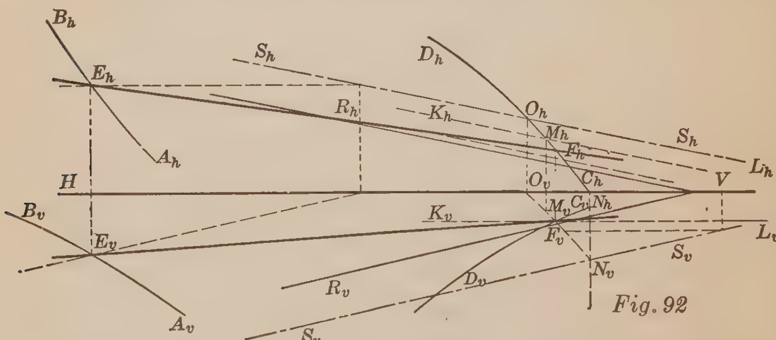



*Fig. 92*

Fig. 92 shows the general solution of this problem. The lines AB and CD are the directrices and R the plane director. S is an auxiliary plane through E, parallel to the plane R. This plane is pierced by the directrix CD, in the point F, and EF is an element of the warped surface. To find the point F, pass a

cylinder through CMD perpendicular to H. This auxiliary cylinder is pierced by the traces of S in the points O and N. The line KL of S pierces the auxiliary cylinder in M. The curve NMO is the intersection of S with the projecting cylinder of CD, and F is the point in which CD pierces the plane S. The required element of the warped surface passes through the points E and F.

139. Second Method. PROBLEM 50. SOLUTION. Since two or more lines are sufficient to determine a plane (Art. 51), if two or more lines of the plane director be drawn, and through any point of one directrix lines parallel to these lines be drawn, they will determine a plane which is parallel to the plane director, and is pierced by the other directrix in a point of the required element.

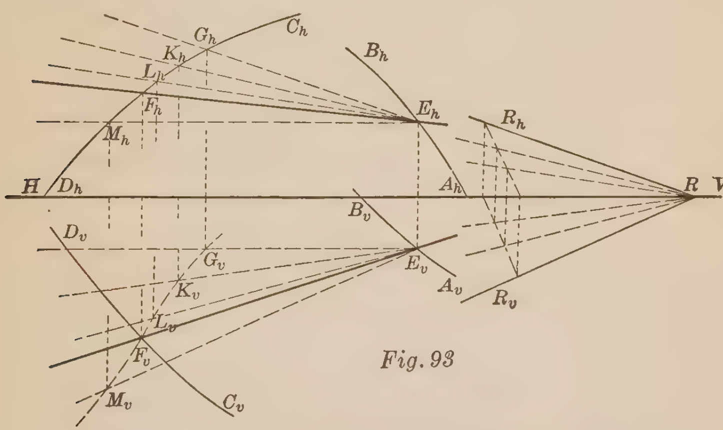



*Fig. 93*

CONSTRUCTION. Fig 93. In the plane R, two lines are drawn which, taken with the traces, make four lines of the plane director intersecting at a point. Through E in AB four lines are drawn, parallel, respectively, to the four lines of R. These lines determine a plane, parallel to the plane director, which contains the required element of the surface, and is pierced by CD in a point of that element. The point E of AB is assumed at will. To find F of CD, draw the horizontal projecting cylinder of CD. The lines of the auxiliary plane pierce this cylinder in the points G, K, L, M. The vertical projection of this curve of intersection is shown at $G_vK_vL_vM_v$, intersecting the vertical projection of CD in the point $F_v$, the point in which CD pierces the auxiliary plane. The required element is the line EF.

140. PROBLEM 51. To draw an element of a warped surface with a plane director which shall be parallel to a given line, the given line being in the plane director or parallel with it.

In Fig. 94 let AB and CD be the directrices, R the plane director, and EF a line of R the given plane.

SOLUTION. If through either directrix, lines be drawn parallel to the given line, they will be elements of a surface which contains the required element of the warped surface. As this element intersects the second directrix, it will pass through the point in which the second directrix pierces the auxiliary surface.

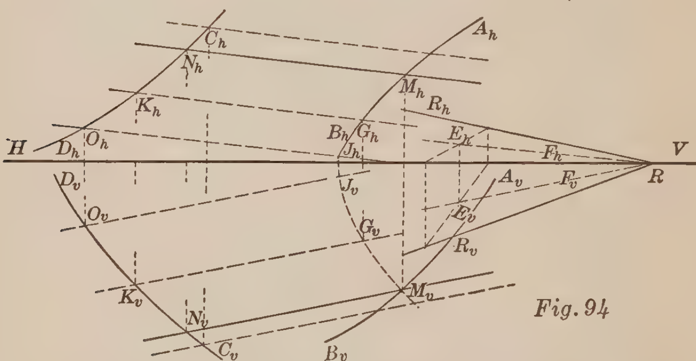



*Fig. 94*

CONSTRUCTION. Through the points K, O, etc., of the line CD, draw lines parallel to the given line EF, of the plane director. Since CD is any curve, these parallel lines will be elements of a cylinder. To find the point in which AB pierces this surface, pass the horizontal projecting cylinder of AB. The lines through K, O, etc., parallel to EF pierce the horizontal projecting cylinder at G, J, etc., points of the intersection of the two surfaces. The vertical projection of the line of intersection intersects $A_vB_v$ in $M_v$, horizontally projected at $M_h$. The point M, being in a cylinder passing through the directrix CD, and also in the directrix AB, the line MN, parallel to the given line EF, will be the required element of the warped surface.

141. PROBLEM 52. To assume a point on a warped surface.

In Fig. 95 the warped surface has AB and CD as directrices and R for its plane director. To assume a point on the surface whose horizontal projection is $M_h$.

SOLUTION. If one projection of the point be assumed, as its horizontal projection, and a plane be passed through the point,

perpendicular to H, it will cut from the warped surface a line which contains the assumed point.

CONSTRUCTION. In Fig. 95 it is necessary to first draw at least three elements of the surface, which may be done by either method of Prob. 50. Through the assumed horizontal projection $M_h$, pass any plane perpendicular to H, as the plane S. This plane is pierced by elements of the surface in the points E, F, G, locating the curve whose vertical projection is the line $E_vF_vG_v$. The vertical projection of the required point is at $M_v$.

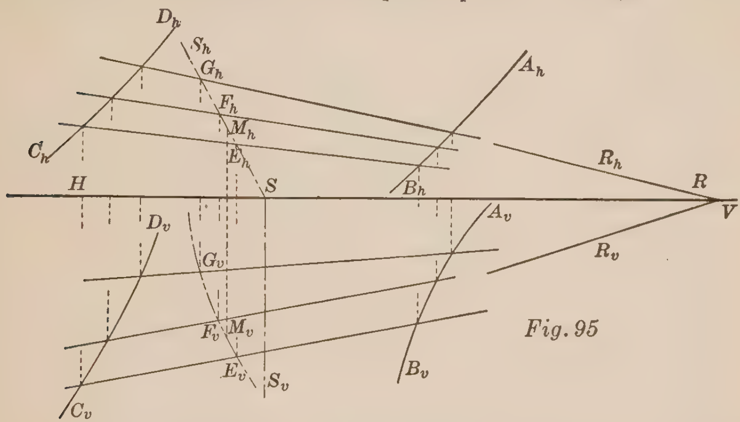


*Fig. 95*

142. It may be shown for warped surfaces as in single and double curved surfaces, that the generatrix may become fixed and the directrices move according to some law, resulting in the regeneration of the surface. In the case of the hyperbolic paraboloid, therefore, any two elements of the surface may be taken as directrices, a plane parallel to the given directrices as the plane director, and the surface be regenerated by moving either of the original directrices, as generatrix, always in contact with the new directrices, parallel to the new plane director.

143. PROBLEM 53. To pass a plane tangent to the hyperbolic paraboloid at a given point on the surface.

SOLUTION. If through the given point an element of each generation be drawn, being lines of the surface, they will be lines of the required tangent plane.

CONSTRUCTION. In Fig. 96 let AB and CD be the directrices and BC and AD the elements of the first generation. Let BC and AD be directrices and AB and CD be elements of the second generation. The assumed point O, on the surface, is located by

Prob. 52 (construction not shown). Draw EO and FO parallel to the elements of the first generation. They will determine a plane parallel to the first plane director, which is pierced by CD at M. Then OM is an element of the surface through the assumed point of tangency and is therefore a line of the required tangent plane. In a similar manner ON is found to be an element of the second generation through O, and is therefore a second line of the tangent plane. The plane of OM and ON is the required tangent plane at O. In the figure the traces of this plane are not drawn.

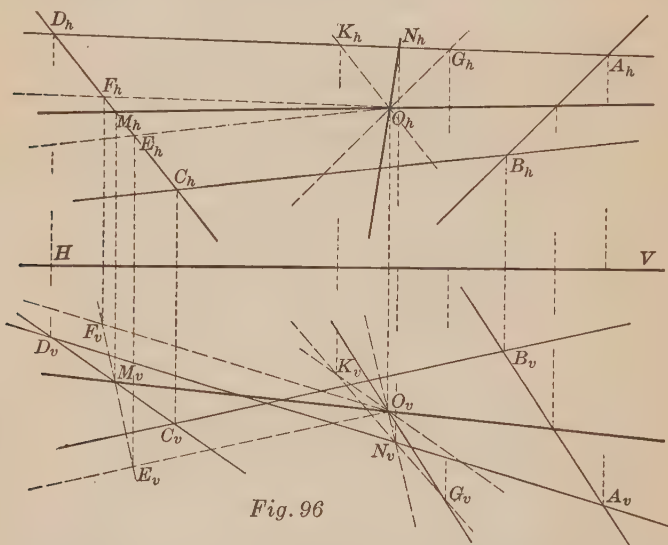



*Fig. 96*

In the construction of this problem the plane director is not shown, but is made use of in a manner now explained. Any two lines are assumed at will as the directrices, as AB and CD, Fig. 96. Draw at will any two lines intersecting these lines as AD and BC. These are elements of the warped surface, parallel to a plane director, for we may pass a plane through any convenient point in space which will be parallel to these assumed elements and would therefore be their plane director. In a similar manner the elements of the second generation have a plane director parallel to AB and CD.

144. In general, a plane tangent to a warped surface at a point may be found by drawing any two lines of the surface passing through that point. Tangents to these lines will lie in the

tangent plane. However, when the surface is of such form that two right lines of the surface may be drawn passing through the point, the tangent plane will pass through them. Surfaces of this class are known as double ruled surfaces.

145. A warped surface may be generated by moving a right line along any two lines, always parallel to the elements of a cone, known as the cone director.

When one directrix is the helix and the other is the axis of the helix, parallel to the axis of the cone director, the surface is the helicoid.

When the cone reduces to a plane (one of its limits) the elements of the warped surface are perpendicular to the right line directrix and it is known as the right helicoid. This may be considered with the warped surfaces with two linear directrices and a plane director. When the elements of the cone director are oblique to the axis, the surface generated is the oblique helicoid.

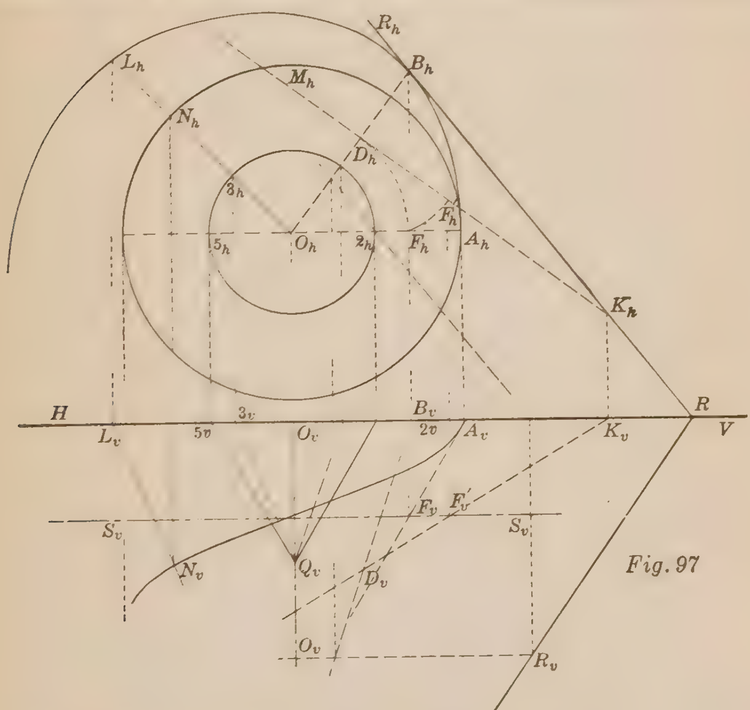



Fig. 97

146. Problem 54. To pass a plane tangent to the helicoid at a given point on the surface.

In Fig. 97 it is required to pass a plane tangent to the helicoid whose base is the curve $A_hB_bL_h$, at the point D.

SOLUTION. The tangent plane will contain the element of the surface passing through the point, and also the tangent to the helix passing through that point.

CONSTRUCTION. To draw the helicoid, we have given the helix AMN and the axis of the helix OO, as directrices, and the cone whose vertex is Q and base 2 3 5 as cone director. Every element of the warped surface always touches the directrices and is parallel to an element of the cone director. Draw the element Q 3 of the cone director. The parallel element of the helicoid will be horizontally projected in $O_hN_h$, as the axis of the cone is also the axis of the helicoid. The vertical projection of the element of the cone is at $Q_v3_v$. The point N, being in the required element, will be vertically projected in the vertical projection of the helix at $N_v$. The required element is therefore vertically projected through $N_v$ parallel to $Q_v3_v$. Produced, this element pierces H at $L_h$, a point in the base of the helicoid. Other elements are drawn in a similar manner.

To pass a plane through D, tangent to the helicoid, draw the element through D, piercing H at $B_h$, a point in the required horizontal trace. The helix through D will be a line of the surface and its tangent at D will be a second line of the required tangent plane. The arc $F_hD_h$, is a portion of the horizontal projection of this helix. Pass the plane S through F, parallel to H. Then the origin of this auxiliary helix will be in the plane S at F, just as the principal helix has its origin in H at A. By the method of Problem 30, draw the tangent DF′, piercing H at $K_h$, a second point in the horizontal trace of the required plane. The vertical trace is found in the usual manner.

147. PROBLEM 55. To pass a plane tangent to the helicoid perpendicular to a given right line.

In Fig. 98 it is required to pass a plane tangent to the helicoid, perpendicular to the line AB.

SOLUTION. Construct an auxiliary cone with its axis coinciding with the axis of the helicoid and whose elements make the same angle with H as is made by the elements of the helicoid. Through the vertex of this auxiliary cone pass a plane perpendicular to the given right line. Then, if the plane fail to intersect the base, the problem is impossible. If it be tangent to the base,

but one plane can be passed tangent to the helicoid perpendicular to the given line: if it intersect the base, two tangent planes may be passed. The element or elements cut from the surface will be parallel to the element or elements of tangency of the helicoid, and they will have the same horizontal projections. Planes through these elements, parallel to the auxiliary plane, will be the required tangent planes.

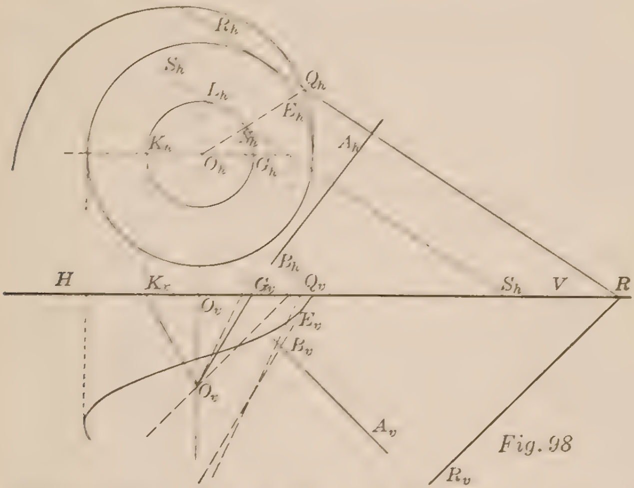


*Fig. 98*

Construction. The helicoid is given by the projections of its elements and the directrices as in Prob. 54. The auxiliary cone used as cone director may be used in the further solution of the problem. Through the vertex of this cone pass the plane S, perpendicular to the given line AB, cutting the elements OL and ON from the auxiliary cone. But the axes of the two surfaces coincide and the horizontal projections of the elements cut from the cone coincide with parallel elements of the helicoid. Producing the element OE, of the helicoid, it pierces H at $Q_h$, a point in the required horizontal trace. Through $Q_h$ pass the plane R, parallel to the plane S. A similar construction locates the plane passing through the element, parallel to OL.

148. In Fig. 98 the plane R passes through the element of the helicoid OE, and is also tangent to the surface. For, if a plane contain an element of a warped surface and is not parallel to its plane director, it will be tangent to the surface at some point along that element. Each of the other elements of the surface will pierce the plane in a point, which joined will give a line

of the surface intersecting the given element. The tangent to this curve will be in the given plane, as all the points of the curve will be in that plane. Then the plane, containing the element of the surface (its own tangent) and a tangent to a curve of the surface, is tangent to the surface at the point where the curve intersects the element.

149. If a right line be revolved about another right line as an axis, at a constant distance from it, and always making the same angle with a plane perpendicular to the axis, the surface generated will be a warped surface known as the hyperboloid of revolution of one nappe.

This is the only warped surface of revolution. As a surface of revolution every point in the generatrix will have for its path the arc of a circle. The path of the point nearest the axis is the circle of the gorge. It is common to assume the axis of the surface perpendicular to H. The base is therefore generated by the revolution of the point in which the directrix pierces H.

This surface belongs to the class of warped surfaces with cone directors, and is a special case in which the directrices are circles parallel to the base of the cone director, with the line through their centers parallel to the axis of the cone. This surface may also be classed with warped surfaces with three linear directrices (Art. 153)—the directrices being circles of different diameters with their centers in the same straight line and their planes perpendicular to this line. The smallest circle, the circle of the gorge, must be placed between the others.

150. Problem 56. To draw the projections of the hyperboloid of revolution of one nappe and to locate a point on the surface.

Solution. Since the surface is generated by a right line revolving at a constant distance from the axis, the horizontal projection of the path of the point nearest the axis will be the circle of the gorge. In all possible positions of the generatrix it will be horizontally projected tangent to the circle of the gorge. Similarly, the base is found by revolving the point in which the generatrix pierces H. As in double curved surfaces of revolution the vertical projection is the meridian curve of the surface parallel to V, or the principal meridian. This may be found by passing through the axis a meridian plane parallel to V, and finding where the generatrix in different positions pierces this

plane. To assume a point on the surface, it is only necessary to draw the projections of an element of the surface.

CONSTRUCTION. In Fig. 99 the axis OO, perpendicular to H, and the generatrix AC, are first assumed. Since AC as a matter of convenience has been assumed parallel to V, its horizontal projection is parallel to the ground line. The shortest distance from the generatrix to the axis is the line $O_hL_h$, perpendicular to $A_hC_h$, and $O_hL_h$ is therefore the radius of the gorge circle. The generatrix AC pierces H at A, and $O_hA_h$ is the radius of the base. To draw the meridian curve whose vertical projection gives the contour of the surface, pass the plane S through the axis, parallel to V. Let the generatrix take any position, as EQ. It will pierce the plane S in a point of the meridian curve required. This position of the generatrix pierces S in N, vertically projected in $N_v$, a point in the curve. Other points may be found in a similar manner and the curves drawn. To assume a point on the surface we first assume its horizontal projection as $K_h$. Draw the horizontal projection of an element through $K_h$, tangent to the circle of the gorge. The vertical projection of this element contains the vertical projection of the point.

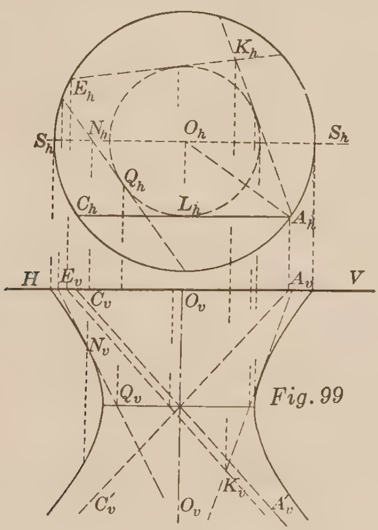


Fig. 99

151. As shown at K, Fig. 99, at any point on the hyperboloid of revolution of one nappe, two lines may be drawn tangent to the gorge circle, it follows that the surface is double ruled, or may be regenerated by another generatrix revolving in an

opposite or the same direction. It should also be noted that two elements of the surface, one of each generation, may be horizontally projected in the same line, tangent to the circle of the gorge. These elements intersect at their point of tangency with the gorge circle, as shown by the two elements vertically projected in $A_v$ $C'_v$ and $A'_v$ $C_v$, intersecting at L on the gorge.

The meridian curve of the surface is the hyperbola, with the two elements of the surface parallel to V as its asymtotes.

152. Problem 57. Pass a plane tangent to the hyperboloid of revolution of one nappe through a given point on the surface.

Solution. If through the given point an element of each generation be drawn, they will determine a tangent plane through the given point.

153. A warped surface may be generated by moving a right line so as to touch three lines. These are known as warped surfaces with three linear directrices. The simplest surface of this class is generated by moving a right line so as to be in contact with three right lines.

154. Problem 58. To draw the projections of an element of a warped surface with three linear directrices which shall pass through a given point on one of them.

In Fig. 100 let AB, CD, EF be the three directrices and let F be the given point.

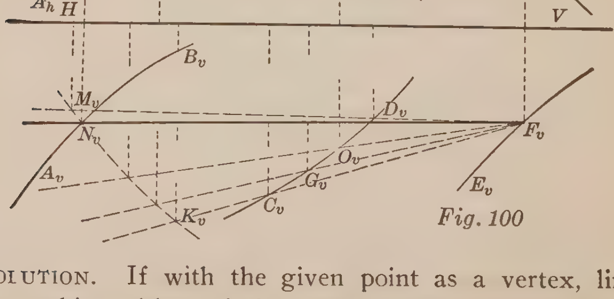


Fig. 100

Solution. If with the given point as a vertex, lines be drawn touching either of the other directrices, they will form

elements of a cone. If the point be found in which this cone is pierced by the third directrix, the line drawn through this point and the given point will be an element of the surface, touching all the directrices. If the cone be pierced more than once by the directrix, more than one element of the surface can be drawn through the given point.

CONSTRUCTION. Draw the elements of the cone FC, FG... FD. To find where this cone is pierced by the line AB, we find the line of intersection of the horizontal projecting cylinder of AB and the cone. This line is vertically projected in the line $K_v N_v M_v$, intersecting $A_v B_v$ in $N_v$. The line FN is the required element of the surface.

If the directrices are right lines the auxiliary cone becomes a plane and it is only necessary to find where this plane is pierced by the third directrix.

Example 1. Construct both projections of the helicoid generated by a line which turns about a vertical axis to which it is inclined at an angle of 45°, one end of the generating line being in the axis. Let the initial position of the line be parallel to V, to the left of the axis, from which position it is to turn about the axis opposite to the hands of the watch, rising 3″ in one complete turn. Show 1 1/2 turns of the helicoid, drawing elements at intervals of 15° or 22 1/2°, and limiting the helicoid by a cylinder of 1 3/4″ radius. Assume a point on the surface and find the traces of the plane tangent to the helicoid at this point. Find also a plane tangent to the surface perpendicular to any assumed line.

Example 2. Find the curve of intersection of the helicoid in Example 1, by a plane passing through a point 2″ above the plane of its base, its horizontal trace making an angle of 30° with HV and the plane making an angle of 30° with H.

# CHAPTER VIII.

## Shades and Shadows.

155. It is sometimes important to know what portion of an object is in the light, what in the shade, and the extent and shape of the shadow which it casts on the plane of its base, or on surrounding objects.

All rays of light are considered parallel right lines when passing through the same medium, or not obstructed or reflected.

Finding the shaded portion of a solid consists in finding the lines of contact with the solid, of either planes of rays or cylinders of rays, or both. These may be tangent planes or tangent cylinders, as in the case of cylinders and spheres; or, planes of rays may only touch the lines of contour, as in the case of rays striking obliquely on all the faces of a cube, touching its edges. That portion of an object between the source of light and the contour line is illuminated. The portion beyond the contour line is in the shade.

In general, the solution of problems in light and shade and the finding of the shadow of an object, either on the planes of projection or on other objects, involves no new principles. The application of the principles given in the chapter on the point, line and plane, and in the chapter on planes tangent to solids, should give the solutions.

When the direction of the rays of light is not given, it is assumed that the light comes over the left shoulder so that the vertical and horizontal projections of a ray make angles of 45° with the HV ground line.

156. Problem 59. To find the shadow of an upright cross upon the plane of its base and upon itself.

Construction. In Fig. 101 let the cross and the direction of rays of light be given as shown by their projections. As the base of an upright figure can not rest on the H plane in the third quadrant, its shadow will not be on H, but on a parallel plane which may be called the plane S, the plane of the shadow. The ray through A pierces S at A′, which is, therefore, the shadow

of A on the plane S. Since the vertical edge through A is perpendicular to S, its shadow will lie in the projection on that plane of the ray through A. Similarly, the shadow of the point C falls at C′. However, since the edge AC is parallel to S, its shadow on that plane is parallel to itself. Similarly, the shadow D′C′ is parallel to CD. The shadow of the vertical edge through A falls on the arm in the line $A_hA'$, and the shaded portion is in the shadow. The shadow of F upon the cross is located by finding where the ray through F pierces the face DKMN in F′. The shadow of GF is parallel to itself. The point O is its own shadow, and $F'O_v$ is the shadow of the edge FO.

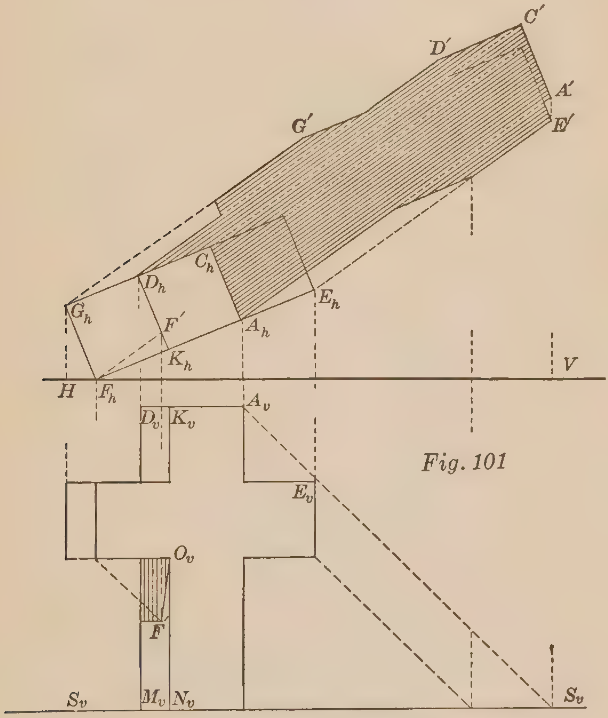


*Fig. 101*

157. Problem 60. To construct the shade of an oblique cone and its shadow upon S.

Construction. Let the cone be given as in Fig. 102 and the projections of the ray of light through the vertex in the line OO′. The shadow of the vertex on S is at O′. All points of the base, being in S, will make their own shadows. Then lines drawn from O′, tangent to the base will be the limiting lines of

the shadow. That portion limited by the elements of tangency of the light, OC and OG, will be in the shade.

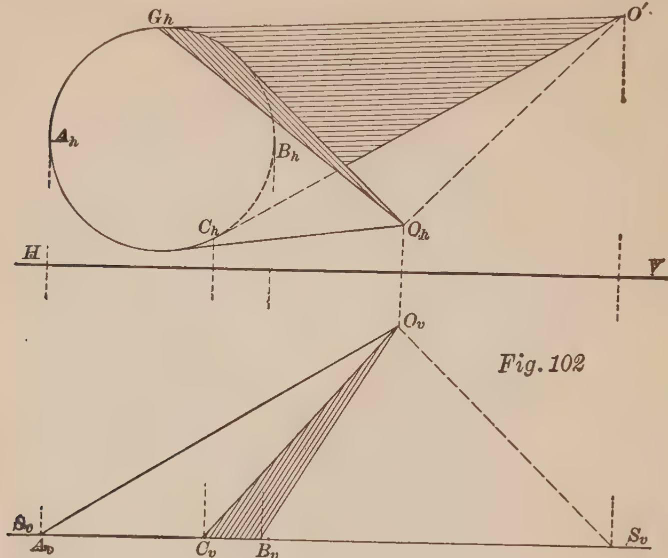



*Fig. 102*

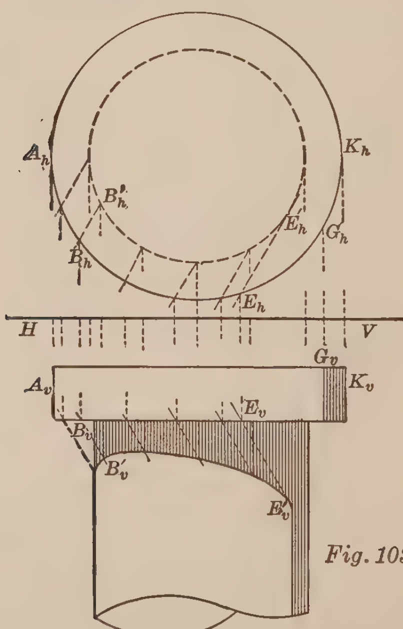



*Fig. 103*

158. PROBLEM 61. To construct the shade of a cylindrical column, and of its cylindrical abacus, and the shadow of the abacus on the column.

CONSTRUCTION. Let the cylinder with its abacus be given as shown in Fig. 103 with the light as shown in the projections of EE′. As the light is tangent to the abacus along the element through G, the shade on the abacus extends from G to K. In like manner the light is tangent to the cylinder along the element through E, and the shade on the cylinder extends from the element through E′ to the right to the contour element. To find the shadow of the abacus on the cylinder, find where the rays through its lower edge pierce the cylinder. Thus the ray through B pierces the cylinder in B′. Other points determine the vertical projection of the curve.

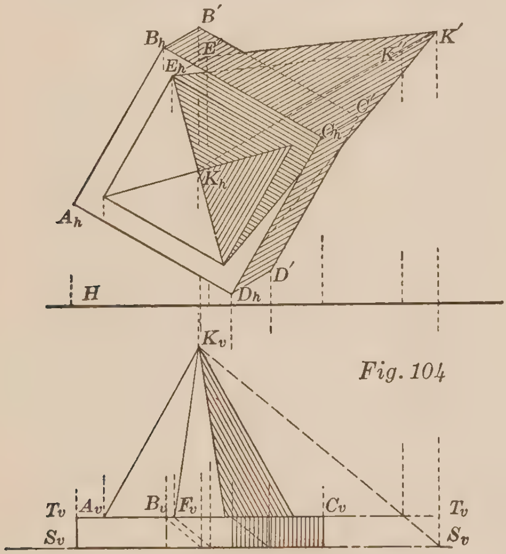


Fig. 104

159. PROBLEM 62. Given the projections of a pyramid resting on its base, to find the shade on the pyramid, its shadow on its base and the shadow of both on the plane of the base.

CONSTRUCTION. The shaded portion of the pyramid is in the shade. The shadow of the base is shown in B′C′, C′D′. The shadow of the vertex K falls at K′. To find the shadow of the edge EK it will be necessary to pass a new plane through the upper base, parallel to H, and find the shadow of K on the auxiliary plane, shown at K″. The point E being in this plane, will be its shadow on that plane. EK″ is therefore the required shadow. To find a second point in the shadow of EK on the S plane, pass a ray through E, its shadow falling at E′. Or KE

might have been produced to pierce the S plane, which point, joined with K′, would give the shadow. Construction is shown in Fig. 104.

160. Problem 63. To construct the shade of an ellipsoid of revolution and its shadow on a horizontal plane, S.

Construction. Let the direction of light and position of ellipsoid be as shown in Fig. 105. The line of contact of the cylinder of rays and the ellipsoid of revolution will be an ellipse. The meridian plane of the ellipsoid, parallel to the direction of light, will also be a meridian plane of the cylinder of rays, and will therefore cut from their curve of intersection, a line of sym-

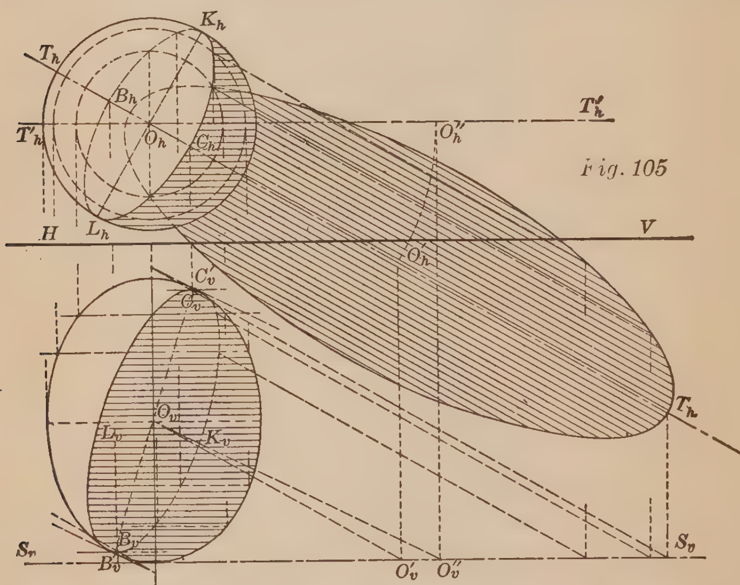



Fig. 105

metry, in this case the major axis of the ellipse of shade. The extremes of this axis are found by revolving both the meridian curve and the rays tangent to it, until they are parallel to V. Thus a ray through O, the center of the ellipsoid, having the direction OO′, when parallel to V will be the line OO″. Parallel to OO″ draw rays tangent to the meridian curve at C′ and B′. After counter revolution the points of tangency are at C and B. The horizontal projection of CB is the minor axis of the horizontal projection of the curve of tangency of light. The minor axis of the curve of shade will be parallel to H and is projected in the line KL. Points in the vertical projection of the curve of

shade may be found by passing auxiliary planes parallel to H and finding where the circles cut from the surface of revolution intersect the horizontal projection of the curve of shade.

Points of the curve of shade may also be found as follows: If a ray of light be projected on any meridian plane, and then a tangent be drawn to the meridian curve, parallel to the projected ray, the point of tangency will be a point in the curve of shade. The meridian plane with the projection of the ray upon it must be revolved parallel to V, and after the point of tangency is found, must be counter-revolved to its original position.

161. A double curved surface, when illuminated, has one or more points which are more brilliant than the surrounding surface. These points have the direction of their surfaces such that the rays of light striking them are reflected directly to the eye, and are known as brilliant points. At such a point, the incident ray, the normal to the surface and the reflected ray are in the same normal plane, and the angle between the incident ray and the reflected ray is bisected by the normal to the surface at the brilliant point.

162. Problem 64. To find the brilliant point on the ellipsoid of revolution, the direction of light and the position of the eye being given.

In Fig. 106 the light is in the direction DO and the line to the eye is horizontally projected in $A_hO_h$.

Solution. The bisector of the angle between a ray of light through any point and the line to the eye through the point is normal to the surface at the brilliant point, which is the point of contact of a plane tangent to the surface perpendicular to the normal. To find this bisector, the plane of the ray and of the line to the eye is revolved parallel to H. The meridian plane containing this normal, cuts a line from the tangent plane which is tangent to the meridian curve at the desired point. Revolving the meridian curve parallel to V, this tangent can be drawn. After counter revolution the point is shown in its true position.

Construction. As the projection is orthographic, any line perpendicular to V may be considered the line to the eye. The plane of DO and AO is revolved parallel to H, and the bisector, parallel to the desired normal, is located at OG. Note that this auxiliary plane is perpendicular to V and that all its lines are therefore vertically projected in $D_vO_v$. The meridian plane S,

through the normal, is revolved parallel to V, the normal being shown after revolution vertically projected in $O_vG''_v$. $M'_vN_v'$ is the line of the tangent plane cut by the meridian plane, tangent to the surface at M′. After counter revolution M′ falls at M, and the brilliant point is located.

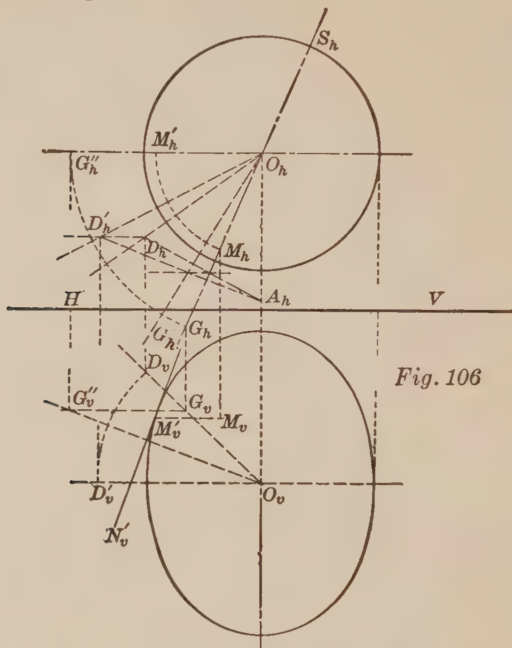


*Fig. 106*

Example 1. Find the brilliant point on a sphere, the line of shade and the shadow on H. The radius of the sphere is 1 1/2″, its center is 2″ above H and 3″ in front of V, and the projections of the rays of light make an angle of 30° with H and 45° with V.

Example 2. An elliptical dome is cut from 12 pieces of sheet metal, as illustrated in the sketch, Fig. 107. The radius of the circumscribed circle of the base is 16′, the height is 20′. It is required to find the element which reflects light to the eye of the observer. The horizontal projection of a ray of light makes an angle of 45° with the ground line and the vertical projection makes an angle of 30° with the ground line, the projection being orthographic.

NOTE: The surface being made up of 12 sheets of metal, will only approximate an ellipsoid. Instead of a brilliant point there will be a brilliant line.

Example 3. Find the brilliant element of a right cone with

a circular base. Assume all dimensions and the direction of light.

Example 4. A hollow cylinder rests on H with its axis parallel to H and making an angle of 30° with V. The outside diameter is 3″, the inside diameter 2 1/2″, length 3 1/2″. Light comes over the left shoulder, the horizontal and vertical projections of rays making angles of 45° with the ground line. Find the lines of shade, the brilliant elements and the shadow on H and on the interior of the surface.

Example 5. Find the brilliant points on the torus, its axis being perpendicular to H. Assume all dimensions and usual direction for light.

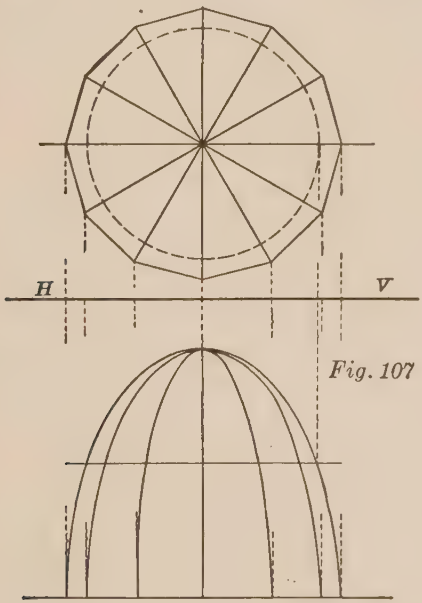



*Fig. 107*

# CHAPTER IX.

## PERSPECTIVE.

163. Since orthographic projections are made on the supposition that the eye of the observer is at an infinite distance, they can never show an object as it actually appears to an observer at a finite distance.

164. Linear perspective has for its object the representation on a single plane, of the general form, and contour lines of a body, and as many of the details of construction as the special problem requires, as they actually appear to the eye of the observer.

165. The vertical plane on which the drawing is made is known as the picture plane, and is usually placed between the object to be drawn and the eye of the observer, with the purpose of making the drawing smaller than the object, and also to prevent the distorted drawings which result from placing the object too close to the eye. When the eye is placed above and a perspective drawing made on an H plane, the picture is known as a bird's eye view. This method is sometimes used to make pictures of large areas, as town sites, exposition grounds, etc.

166. As in orthographic projection, the projection of a point is found by drawing a line from the eye to the given point, and finding where this line, produced if necessary, pierces the plane of projection, or picture plane. This projection, however, is known as the perspective of the point.

167. The eye is known as the point of sight. Lines drawn from the point of sight are known as visual rays. If drawn to touch a right line they form a visual plane; if to touch a curved line, when the eye is not in the plane of the curve, or a curved surface, they form a visual cone.

As the projection of the eye, or point of sight upon the picture plane is a frequent point of reference, it is known as the principal point of the picture.

Since the principal point is the orthographic projection of the eye upon the picture plane, a horizontal line passing through the principal point is as high as the eye and is known as the horizon of the picture.

168. Problem 65. Find the perspective of a point when the orthographic projections of the eye and the point are given.

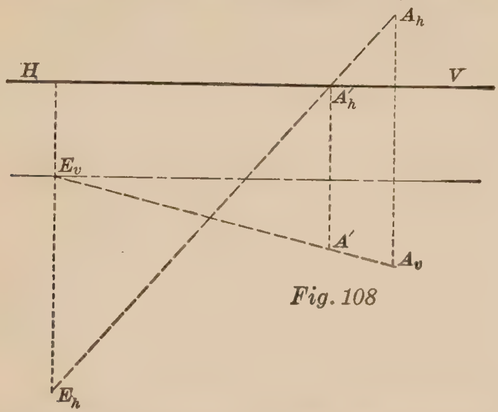



Fig. 108

In Fig. 108 let E be the eye of the observer and A the given point. Then $E_v$ is the principal point of the picture and the horizontal line through $E_v$ is the horizon. Drawing a line from the eye to the point, it pierces V in the point A′, which is the perspective of the point A.

This construction is absolutely general and may be made to cover every case; for the perspective of a right line may be determined by two of its points, curved lines by successive points, planes and surfaces by lines, etc.

169. It is possible by taking advantage of rules which apply to classes of points and lines, to shorten the number of operations necessary in finding their perspectives by the general method given for finding the perspective of a point.

170. Problem 66. Find the perspective of a line when its orthographic projections and the projections of the point of sight are given.

In Fig. 109, $E_v$ is the principal point and AB the given line.

Solution. If visual rays be drawn through consecutive points of the given line and the points be found in which they pierce the vertical plane, the line joining these points will be

perspective of the given line. But visual lines drawn to touch a right line form a visual plane and they pierce V in the vertical trace of the plane. Therefore, if through the eye and the given line a visual plane be passed, its vertical trace will be the perspective of the line. The point in which the given line pierces the plane of the picture will be one point in the required perspective; or, in general, if a point is in V, it will be its own perspective. To find another point of the perspective of the line, or another point in the vertical trace of the visual plane through the line, draw a line through the eye, parallel to the given line. This will be a line of the visual plane and will pierce V in a point of its vertical trace, and is therefore a second point in the perspective of the line AB.

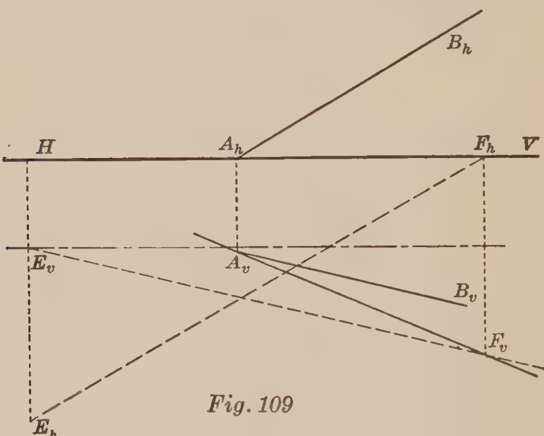



*Fig. 109*

But the line through the eye, parallel to the given line, will be parallel to all lines parallel to that line. Then the point in which it pierces the picture plane will be a point common to the perspectives of all lines parallel to the given line, and is known as the vanishing point of those lines.

Therefore, to find the vanishing point of a system of parallel lines, draw a line through the eye parallel to the system of lines, and find where it pierces the picture plane. As parallel lines are assumed to meet in a point at an infinite distance, the vanishing point of a system of parallel lines is the perspective of this their assumed meeting point.

Joining the point in which a given right line pierces the picture plane and the vanishing point, with a right line, determines the perspective of the line.

If the given line be parallel to H, the line from the eye, parallel to the given line, will pierce V at a point as high above V as the eye of the observer, or as high as the principal point of the picture, therefore in the horizon. The vanishing point of all lines parallel to H will therefore be in the horizon.

The vanishing point for all lines perpendicular to the picture plane will be at the principal point of the picture.

Ex. 1. Draw the perspective of a 4″ cube, one edge being in the V plane, the base being 8 inches below H, the eye eight inches in front of V, one inch below H, and two inches to the right of the right-hand edge of the cube. Let an edge of the base make an angle of 30° with the ground line.

Ex. 2. Draw the perspective of a square pyramid resting on a square base. Place the eye well above and to the right of the object. Make use of vanishing point of parallel lines.

Problem 67. Let it be required to find the perspective of a point by the method of perpendiculars and diagonals.

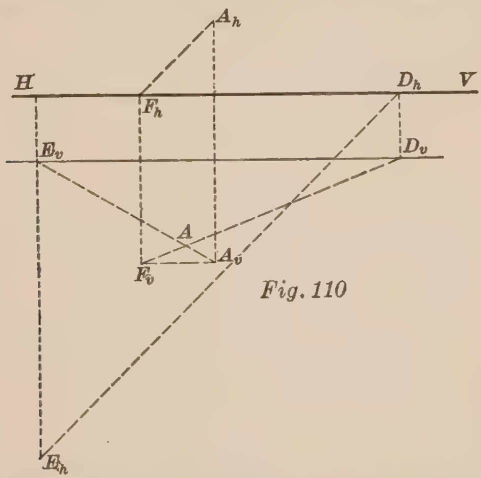



*Fig. 110*

In Fig. 110 the perpendicular pierces V at $A_v$. Then $A_vE_v$ is the perspective of the perpendicular. The vanishing point of a line making an angle of 45° to the right of a perpendicular is at $D_v$, known as the vanishing point for diagonals. A line through A, parallel to ED, pierces V at $F_v$, one point of the perspective of the diagonal. The vanishing point for this line being at $D_v$, the perspective of the line is $F_vD_v$. Since the diagonal and perpendicular were drawn through A, their perspectives intersect in the perspective of the point at A′.

173. The diagonal, making an angle of 45° with the perpendicular to V, will, if drawn through the eye, vanish in the horizon at a distance from the principal point equal to the distance of the eye of the observer from the picture plane. The vanishing points for diagonals are therefore known as distance points.

174. As in orthographic projection, if a line be parallel to the picture plane, its perspective is parallel to the line itself. Then to draw the perspective of lines which are parallel to the picture plane, it is only necessary to find the perspective of one point of the line and through that point draw a line parallel to the vertical projection of the given line.

175. The perspective of curves may be found by determining successive points and joining them by a curve.

If the plane of the curve be parallel to the picture plane, its perspective will be a similar curve. Thus if the curve be a circle, its perspective will be a circle. In all positions except with its plane parallel to the picture plane and with its plane in the horizon plane, the perspective of a circle will be an ellipse.

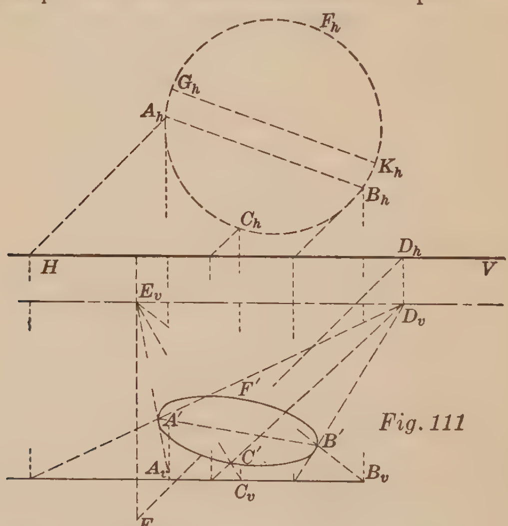



*Fig. 111*

176. Problem 68. To find the perspective of a circle. In Fig. 111 the plane of the circle is parallel to H. $E_v$ is the principal point of the picture and the vanishing point for perpendiculars, $D_v$ is the distance point, $E_vD_v$ the horizon and $F_h$ the point of sight.

By means of the perspectives of perpendiculars and diagonals the points A′ and C′, etc., are found. The greatest angle of vision is obtained when tangents are drawn from the point of sight to the circle. The perspectives of the points of tangency are shown at A′ and B′, and A′B′ will be the major axis of the ellipse. Additional points may be determined in like manner and the curve drawn through them.

It should be noted that the major axis of the ellipse is not the perspective of a diameter as of GK, but of a chord, AB, parallel to GK. Moreover, A and B are the points of tangency of visual rays, and if the figure were a solid, would be points in the line of apparent contour.

177. As rays of light are considered as parallel lines, they will have a vanishing point which may be found by drawing the projections of a ray through the eye, and finding where it pierces V.

The direction of the light is assumed at will, either by drawing the projection of a ray, or assuming the vanishing points of rays. The draftsman is more certain to place the shadow of an object where he wants it by assuming the vanishing point for rays, and this method is usually followed.

178. Since it is usually required to find the shadow of the object on a horizontal plane passing through its base, it is necessary to find the perspective of the point in which a ray pierces this horizontal plane. This may be found by the general method of the perspective of a point, but the usual method is to find the perspective of the ray and the perspective of its projection on the plane of its base. The intersection of these perspectives will be the required point.

The projection of a ray, on the plane of the base, will be a horizontal line and will therefore vanish in the horizon.

A ray and its projection on the base plane being in the same plane perpendicular to H, their vanishing points will be found in the same perpendicular to the ground line. Therefore, if the vanishing point for rays has been assumed or found, a perpendicular to the ground line through this point intersects the horizon in the vanishing point for the projection of rays on the plane of the base.

179. PROBLEM 69. Construct the perspective of a rectangu-

lar pillar resting on a base, with its shade, and shadow on the plane of its base.

CONSTRUCTION. The figure 112 shows the perspective found in part by the method of vanishing points for parallel lines and in part by the general method of the perspective of a point. The student is at liberty to use these or the method of perpendiculars and diagonals if he prefers. All lines of the object parallel to AB will have their perspectives intersecting in the vanishing point $L_v$, and one point in each line is sufficient. The vanishing point for rays is assumed at $M_v$. The vanishing point for the rays on the plane of the base will be in the horizon at $N_v$, in the perpendicular to the ground line through $M_v$, the vanishing point for rays.

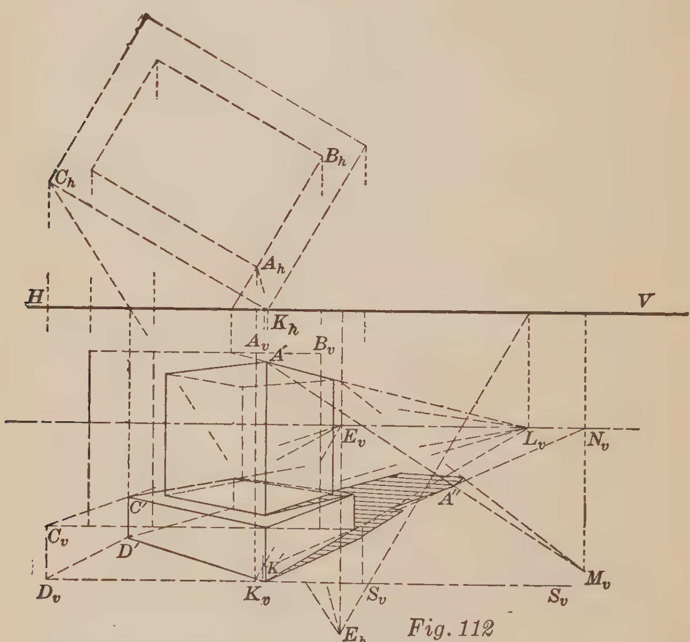


Fig. 112

To find the perspective of the shadow of any point as A, through the perspective of the point draw the perspective of a ray, as the line $A'M_v$. The point A is projected on the plane of the base in K, shown in perspective at K′. $K'N_v$ is therefore the perspective of the projection of the ray on the plane of the base. The intersection of these perspectives gives the perspective of the shadow of the point, A″.

180. PROBLEM 70. To construct the perspective of a cylindrical column with a square pedestal and abacus, and the shade of the column and shadow of the abacus on the column.

CONSTRUCTION. The projections of the column with its base and abacus are shown in Fig. 113. As one face of the abacus and one face of the base are placed in the vertical plane, or picture plane, they will be their own perspectives. The eye is projected on the picture plane at $E_v$, the principal point. The eye is placed at a distance in front of V equal to the distance $E_vD_v$. The horizontal projection of the eye is not shown but could be found by laying off from the ground line along the perpendicular through $E_v$ the distance $E_vD_v$. The perspectives of the various points are found by the method of perpendiculars and diagonals, though the student may use any method he may prefer. The

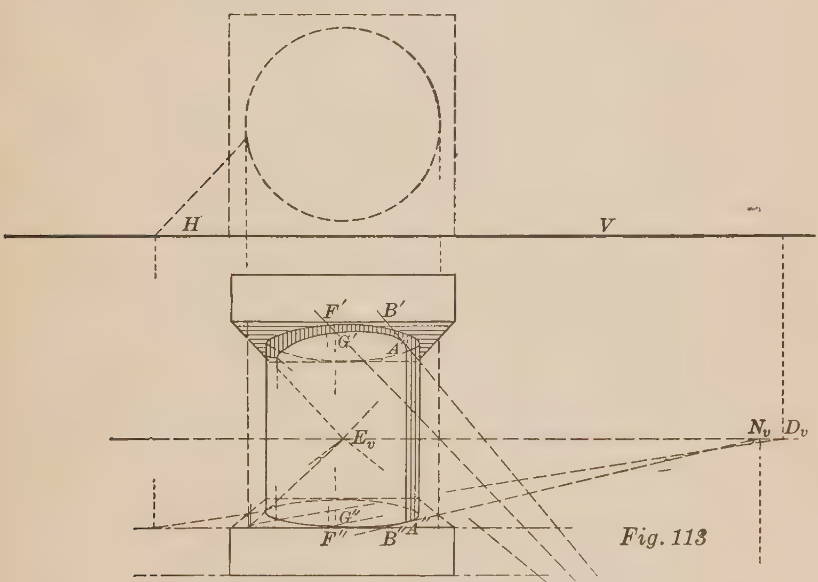


Fig. 113

vanishing point for rays is assumed at $M_v$ (not shown in the drawing), giving the vanishing point for the rays on the plane of the base in the horizon at $N_v$. The element along which light is tangent is found by drawing the perspective of the projection of a ray on the plane of the base, tangent to the base of the cylinder at A″. The ray which has the perspective of its projection tangent to the cylinder at A′, passes through the point B′, on the lower edge of the abacus, determining a point of its shade on the

cylinder. Other points of this curve may be found by first drawing the perspective of the projection of a ray on the base plane, as through F″, cutting the cylinder at G″. Draw the element through G″. The perspective of the ray through F′ intersects this element in the shadow of G′ on the cylinder. Other points are found in a similar manner.

Example 1. Construct the perspective of a hollow cylinder with its axis perpendicular to H, and show its shade and shadow on its interior surface.

Example 2. Show the perspective of the frustum of an inverted cone, with its axis perpendicular to H, and show its shade and shadow.

Example 3. Show the perspective of a square pyramid resting on a square base, with all the edges of the base oblique to the picture plane, the eye being to the right of the projections of the object. Show the shade and shadow on the base and the shadow on the plane of the base.

Example 4. Show the perspective of a cross, the portion in the shade and the shadow on the arm and on the plane of the base. The cross has the following dimensions: Height to arms 6″, timber of cross 1 1/2″, length of arms 3″, height above arms 3 1/4″. The picture plane contains an edge of one of the arms of the cross, the horizontal projection of the arm making an angle of 15° with the picture plane. The eye is 10″ from V, 3″ to the right of the center of the cross, and midway between the base and the top of the arms.

Example 5. Assume the plan and elevation for a cottage of simple design. Select the best point of sight with reference to the picture plane and make a complete perspective drawing.

## EXAMPLES.

### THE POINT, LINE AND PLANE.

Ex. 1. Given the traces of two planes, (0.25, 0, 0), (2.25, 0,—2), (1.75,—1.5, 0); (3.75, 0, 0), (2.25, 0,—2), (1.75,—1.5, 0), find the line of intersection and the true length of that portion of it which is included between H and V.

Ex. 2. Find the traces of the plane which passes through the points (6,—1,—1), (8,—0.5,—2) and (9,—2,—2). Construct the projections of a square situated in this plane, the horizontal projection of one of its diagonals being (4.5, 1, 0) (7, 2, 0).

Ex. 3. Given A(6.5—2.25,—1.25) B(8.5,—3.5,—2.5); pass a plane through AB so that the vertical trace may make an angle of 45° with the ground line. Construct the projections of a cube which stands on the plane, AB being the edge nearest the horizontal trace. Draw the line C(3.5,—1.25,—5.0) D(9.5,—3.5,—2.75) and find the points where it intersects the cube.

Ex. 4. A line passes through the points (0, 3, 3), and (4, 1.75, 0); and another through the points (2.75,—0.25, 0), and (4.75, 1.75, 2.5). Find the shortest line which can be drawn joining the two given lines, and construct a parallelopiped, three of whose edges are the given lines and the perpendicular between them; making the edges all equal to the perpendicular, and placing the solid *above* and *in front* of it.

Ex. 5. Pass a plane through A(5.75,—2,—1.75), perpendicular to B(10,—2.75,—2.5), C(11.5,—0.75, 0). Pass a second plane through A parallel to the same line and also parallel to the line D(10,—2,—1.9), E(11.5,—1,—1.4). The first plane forms the lower face of a cube, the second plane forms another face, and A is that vertex of the lower face which is farthest from H and farthest to the right. The edge of the cube is 2 inches; find its projections.

Call the vertices of the lower face A, F, G, and K, and those of the upper face A′, F′, G′, K′. (AF is the edge formed by the intersection of the two given planes.) Pass a plane through the middle points of AF, AK and KK′ and find the section of the cube by this plane.

Revolve the section upon H to the right in such a manner as to show the true form of the section without interfering with the remainder of the problem.

Ex. 6. The point A(5.75,—2,—2) is the center of a cube whose edge is 3 inches, one face is parallel to the plane (0, 0, 0) (3, 0,—1.5) (3,—1.75, 0), and the vertical projection of the perpendicular from A upon another face passes through B(7.75, Y, —3.75). Construct the projections of the three perpendiculars through A to the faces of the cube.

Ex. 7. Given the lines A(2,—1,—4), B(6.2,—3.4,—1.25), and A (2,—1,—4), C(6.2, 3.4, 2.5). A perpendicular to AB is erected at B meeting AC produced, and forming a right triangle in space. This triangle revolves about AB and it is required to draw the projections of eight of its positions at equal intervals of 45°. To do this, pass a plane through B, perpendicular to the axis. The base of the triangle describes a circle in the plane.

Revolve B upon H, draw the circles, and determine the revolved positions of the moving angle of the triangle, then find the true positions on the plane.

Ex. 8. From the point, (7,—1.5,—3.5), let fall a perpendicular upon the plane (6, 0, 0), (0, 0,—3.5), (10,—4, 0). Construct the projections of a regular tetrahedron of which the perpendicular shall be the altitude, one base being in the given plane, and one of the sides of this base (the one farthest to the left), parallel to a line in the given plane, whose horizontal projection is (2, 0, 0), (0,—3, 0).

Construction. Revolve the foot of the perpendicular upon H to the right; construct the base of the tetrahedron in that position and revolve it back. The ratio of the altitude of a tetrahedron to its edge may be found by constructing the projections of one at a convenient place on the paper, in this take one side perpendicular to the ground line and, say, 1.5 inches in length.

Ex. 9. The vertices of a quadrangular pyramid are given as follows: E(0.5,—0.5,—3.5), A(3,—2.5,—1.25), B(4,—1.5,—0.75), C(3.5,—0.5, 0), D(2.5,—1.0, z). The vertex is the point E. It is required to find the third co-ordinate of the point D, to construct the pyramid, and to develop its surface.

Make the development by revolving the face EAB to the right about its horizontal trace until it coincides with H, attach the other faces in their order working upwards, and join the base to the face EAD.

Ex. 10. It is required to construct a truncated prism with its base in H and is terminated by V; the upper base is a quadrilateral whose vertices are A(5.5,—3.5, 0), B(6,—2, 0), C(7.5,—1.5, 0) and D(8,—2, 0). One edge passes through A and A′(9.5, 0,—4). Having constructed the prism, develop its surface by revolving the face which meets H in AB to the left until it coincides with H; to this face attach the lower base and then the other faces in their order, working towards the left.

## Intersections of Solids by Planes.

Ex. 1. A cube whose edge is 1.5 has its upper vertex at (2, 2, 3); the diagonal through this vertex is vertical and the cube is so situated that a plane passed through this diagonal, perpendicular to V, contains two of the edges.

1. Construct the projections of the cube in the given position.

2. Suppose the solid moved 3 inches to the right, and then to be turned through an angle of 45° about the vertical diagonal as an axis; the direction of rotation being opposite to that of the hands of a watch. Construct the projections of the cube in its new position.

3. Having constructed the cube in its last position, pass a plane through the middle points of the two edges of the *upper* face which lies farthest to the right, and also through the middle point of another edge, so that the section may be a regular hexagon. Finally revolve the section upon V to the right.

Ex. 2. A right circular cone lies with an element in H, its vertex at (6,—2, 0) and the upper element parallel to V. The altitude is 5 inches, radius of base 2 inches, and the base lies to the left of the vertex.

1. Construct the projections of the cone, drawing elements at intervals of 15°.

2. Find the horizontal projection of the section of the cone by a horizontal plane 1 inch below H.

3. Find the vertical projection of the section by a vertical plane 3 inches behind V. To construct the right branch of the hyperbola produce the elements in the vertical projection, the vertex is then the *center of symmetry* of the hyperbola.

Ex. 3. A right circular cylinder is in the third quadrant, with its base in H, center of base at (2.25,—2, 0), radius 1.5, altitude 4. A plane perpendicular to V and making an angle of 45°

with H, passes through (4.25, 0, 0) and cuts the cylinder below H.

1. Draw elements of the cylinder at intervals of 15°, and revolve the section upon H to the right.

2. Develop the curve of intersection. To do this suppose the cylinder moved until the extreme *front* element comes into contact with V at (4.25, 0, 0), and then *rolled* along V to the right until it has been turned through an angle of 270°.

Ex. 4. A torus is generated by a circle, radius 1.25 inches, which revolves about a vertical axis through (6.5,—2.5, 0), the initial position of the center is at (8.5,—2.5,—1.25). A plane whose horizontal trace makes an angle of 135° with the ground line, is inclined to H at 45° and is tangent to the *front inner* surface of the torus.

It is required to construct the projection of the torus and the traces of the plane and to find the projections of the curve of intersection. To determine points of the curve pass a series of parallel planes at intervals of 15° on the circumference of the generating circle.

Ex. 5. A right circular cone stands in H, center of base at (4,—2, 0), radius 2 inches, vertex (4, 0,—4.25). It is required to find the section of the cone by a plane whose horizontal trace is (1, 0, 0), (4,—4, 0) and which makes an angle with H equal to that made by the elements of the cone.

Revolve the section upon H to the right, first moving it to the right and turning it far enough to clear the ground line and the rest of the drawing.

Ex. 6. An oblique cylinder with a circular base stands in H, center of base (2 1/2,—3, 0), radius 1 1/4 inches. The horizontal and vertical projections of the elements make angles of 30° and 45° with the ground line, respectively.

It is required to find the projections and true form of a right section by a plane through (4 1/2,—2,—3). (Revolve to the right about the horizontal trace.) Draw tangents to the curve from some point on the horizontal trace of the plane.

Ex. 7. An oblique cone with a circular base stands on H, the center of the base is (5,—2.5, 0), radius 1.5 inches, and the vertex is (8,—1,—4).

It is required to draw the projections of the cone, the projections of its section by the plane (5, 0, 0), (2, 0,—4), (8,—3, 0),

and to find the true form of the section by revolving it to the right upon H.

In the finished drawing show only the limiting elements of the cone, and draw through some convenient point on the trace two tangents to the section.

Ex. 8. A right circular cone stands in H, center of base at (3.0,—2.5, 0), radius 2″, altitude 4″. A plane passes through (4.5, 0, 0), perpendicular to V and parallel to the extreme right-hand element of the cone. Draw elements at intervals of 30° and find the projections of the section (a parabola). Draw a tangent to the curve. Revolve the curve and tangent on V.

Ex. 9. A right circular cone stands on H, center of base at (11.0,—3.0, 0), radius 2.5″, altitude 3.25″. The cone has two nappes, the upper one to be taken equal to the lower one. A plane passes through (13.25, 0, 0) perpendicular to V, and at 60° with H. Find the projections of the section (an hyperbola) and its asymptotes. Revolve the curve and asymptotes upon V, turning the vertical trace until it is perpendicular to the ground line.

### Intersection of Solids.

Ex. 1. A triangular pryamid, K(4,—1.25,—3.5), A(2.75, —0.5,—1), B(3.5,—2.5,—0.75), C(6.25,—1.5,—1.5), is pierced by a *right* triangular prism, one of whose edges is D(1.5,—0.25, —3.25), E(8,—3,—2), the other two edges passing through the points F(2.25,—1.5,—1.75) and G(4.50, 0,—2.25), respectively.

Draw the projections of the solids, showing the lines of intersection.

Ex. 2. A prism has its base in H, both projections of its elements making angles of 30° with the ground line. The base is a regular hexagon, the center being at (2,—3.5, 0), and one vertex at (3,—2.5, 0).

A second prism has its base in H, both projections of its elements making angles of 135° with the ground line. The base is an equilateral triangle, the center being at (6.5,—3.5, 0), and one vertex at (7.5,—4, 0). Construct the projections of the prisms and of the lines of intersection. Use a series of auxiliary planes parallel to the elements of both prisms.

Ex. 3. A torus is generated by a circle which revolves about a vertical axis through (4.5,—3.5, 0). The radius of the generating circle is 1 inch; distance of center below H, 2 inches; distance of center from axis, 2.5 inches.

A cylinder with a circular base stands in H, center of base at (1,—3.5, 0), radius 1 inch; the horizontal and vertical projections of the elements make angles of 45° and 315° with the ground line, respectively.

It is required to construct the projections of the two surfaces, and of the curve of intersection.

Ex. 4. A cylinder, circular base, stands on H, center of base at (4.5,—2.75, 0), radius 1.25″, the elements being parallel to (5.5,—0.75,—3.5), (0.75,—4, 0).

A cone with a circular base also stands on H, center of base at (8.5,—2.25, 0), radius 2″, vertex at (5.5,—0.75,—3.5). It is required to construct the projections of the two surfaces and of the curve of intersection and to draw a tangent to the curve at some convenient point. (The tangent is the intersection of the tangent planes to the two surfaces at the assumed point.)

The finished drawing should show all the elements which *limit* the curve; if the points of the curve lying on the limiting elements are accurately determined, but few others will be required.

## Warped Surfaces.

Ex. 1. A line AC is 4 inches in length; the point A moves in a straight line, parallel to the ground line, through (4,—2.25, 0), while the middle point of AC moves in a straight line perpendicular to H at (4,—1.25, 0).

1. Find the projections of the locus of C.

2. Find the vertical trace of the surface generated by the indefinite line AC.

3. Find the form of a section of the surface made by a plane perpendicular to H and V through (5, 0, 0). Show this by revolving the section to the right upon H about a line perpendicular to the ground line at (7, 0, 0).

Draw eight elements of the surface on each side of the vertical directrix above H, so as to intercept nearly equal distances on the curve which forms the vertical trace of the surface.

Ex. 2. A line ABC is 4½ inches long, AB being 2.75 inches; the point A moves on the circumference of a circle in H, whose center is at (2,—2, 0), and radius 1 inch, while the point B moves on a vertical axis through (3.75,—2, 0). It is required to find the projections of the locus of C, and of the surface generated by the indefinite line AC. Find also the form of a section of the surface by a plane perpendicular to H and V through the

center of the curve which is the horizontal projection of the locus of C. (Revolve the section upon H to the right.)

Draw elements of the surface at intervals of 15° on the circumference of the circular directrix, and producc the vertical projections to the margin of the paper.

Ex. 3. A circle lies in H, center at (4,—2, 0), radius 2 inches, and an equal circle lies in a horizontal plane 4 1/4 inches below the first, the centers of the two being in one vertical line. A line touches both circumferences and moves around them, making a constant angle with H, thus generating *an hyperboloid of one nappe*. The initial position of the generatrix is (4,—4, 0), ($x$,— 1,—4). Draw elements of both generations at intervals of 30° on the directrices, and make a section of the surface by a plane through (5.5, 0, 0), the horizontal trace making an angle of 300° with the ground line, and the vertical trace an angle of 150°. Revolve the curve of intersection upon H to the right.

Ex. 4. A line moves on two directrices, the first is parallel to the ground line and passes through (5, 0, 1), and the second is perpendicular to H at (5,—1, 0). The initial position of the line is (5,—1, 1), (7, 0, 1), and it moves in such a manner that while its intercept on the horizontal directrix *increases*, the intercept on the vertical directrix *decreases half as much*.

Construct the projections of the surface so generated, drawing elements at intervals of 1/2 inch on the horizontal directrix, and at half this interval near the vertex of the *profile curve*. Find the trace of the surface on H.